Quiz 7 **Solutions**, Math 112, Section 2 (Vinroot)

Determine whether each of the following series converges or diverges. Explain the test you are using carefully.

(a): $\sum_{n=1}^{\infty} \frac{\cos^4(n)}{n\sqrt{n}}$.

Solution: We know that $-1 \leq \cos(n) \leq 1$ for any n, and so $0 \leq \cos^4(n) \leq 1$ for any n. This means we have

$$\frac{\cos^4(n)}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

By the p-test, with $p = 3/2 > 1$, we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. By the Comparison Test, then, we can say that $\sum_{n=1}^{\infty} \frac{\cos^4(n)}{n\sqrt{n}}$ also converges.

(b): $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

Solution: Consider the function $f(x) = (\ln x)/x^2$. Then $f(x)$ is continuous and $f(x) \geq 0$ when $x \geq 1$. To use the Integral Test, we need that $f(x)$ is sufficiently decreasing as well. To check this, we find $f'(x) = (1 - 2\ln x)/x^3$, and we see that $f'(x) < 0$ when $x \geq 2$ (or, more precisely, when $x > e^{1/2}$, but all that matters is that it is decreasing from some point onward). So, by the Integral Test, our series converges if and only if the following improper integral converges:

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx.$$

We compute this integral as follows, where we use integration by parts with $u = \ln x$ and $dv = x^{-2}\, dx$, so $du = (1/x)\, dx$ and $v = (-1/x)$. So:

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \left(\left[-\frac{\ln x}{x} \right]_1^t + \int_1^t x^{-2}\, dx \right)$$

$$= \lim_{t\to\infty} \left(-\frac{\ln t}{t} - \frac{1}{t} + 1 \right) = \lim_{t\to\infty} \left(-\frac{\ln t}{t} \right) - \lim_{r\to\infty} \frac{1}{t} + 1 = -\lim_{t\to\infty} \frac{\ln t}{t} + 1,$$

since $\lim_{t\to\infty} (1/t) = 0$. To compute $\lim_{t\to\infty} (\ln t)/t$, we may use L'Hospital's rule, since the limit is of the indeterminate form of type "$\frac{\infty}{\infty}$". So $\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0$. Since the integral converges to 1, then by the Integral Test, the series also must converge.