Quiz 6 **Solutions**, Math 112, Section 2 (Vinroot)

(a): Consider the sequence defined by $a_n = \sqrt[3]{\dfrac{8n^2+2}{n^2+3}}$. Either compute the value of the limit $\lim_{n\to\infty} a_n$, or show that it diverges.

Solution: We apply the limit laws of sequences as follows:

$$\lim_{n\to\infty} \sqrt[3]{\frac{8n^2+2}{n^2+3}} = \sqrt[3]{\lim_{n\to\infty} \frac{8n^2+2}{n^2+3}} = \sqrt[3]{\lim_{n\to\infty} \frac{(8n^2+2)(1/n^2)}{(n^2+3)(1/n^2)}}$$

$$= \sqrt[3]{\lim_{n\to\infty} \frac{8+\frac{2}{n^2}}{1+\frac{3}{n^2}}} = \sqrt[3]{\frac{\lim_{n\to\infty} 8 + \lim_{n\to\infty} \frac{2}{n^2}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{3}{n^2}}}$$

$$= \sqrt[3]{\frac{8+0}{1+0}} = \sqrt[3]{8} = 2.$$

(b): Either compute the sum of the following series, or explain why it diverges:

$$\sum_{n=1}^{\infty} 3^{n+1} 2^{-2n}.$$

Solution: We attempt to recognize the series as a geometric series, since there are nth powers. We have:

$$3^{n+1}2^{-2n} = 3(3^n)(2^{-2})^n = 3\left(\frac{3}{4}\right)^n = \frac{9}{4}\left(\frac{3}{4}\right)^{n-1}.$$

That is, we can rewrite the series as

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \sum_{n=1}^{\infty} \frac{9}{4}\left(\frac{3}{4}\right)^{n-1},$$

which is a geometric series with first term $a = 9/4$ and ratio $r = 3/4$. Since $|r| = 3/4 < 1$, then the series converges, and is equal to $a/(1-r)$. So, the series converges and is equal to

$$\sum_{n=1}^{\infty} \frac{9}{4}\left(\frac{3}{4}\right)^{n-1} = \frac{9/4}{1-(3/4)} = \frac{9/4}{1/4} = 9.$$