Quiz 2 **Solutions**, Math 111, Section 1 (Vinroot)

(a): Compute the following limit (if it exists), making your steps as clear as possible: $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x+3x^4}}{x^2-3}$

Solution: The exponent of the denominator is 2, which means we should multiply the numerator and denominator by $1/x^2$. In the numerator, so that we may bring this factor under the square root, we rewrite it as $1/x^2 = \sqrt{1/x^4}$, noting that all of the terms are positive. Doing this, we have

$$\lim_{x\to\infty} \frac{\sqrt{x+3x^4}}{x^2-3} = \lim_{x\to\infty} \frac{\sqrt{x+3x^4}}{x^2-3}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt{x+3x^4}}{x^2-3}\cdot\frac{\sqrt{1/x^4}}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt{\frac{x}{x^4}+\frac{3x^4}{x^4}}}{\frac{x^2}{x^2}-\frac{3}{x^2}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^3}+3}}{1-\frac{3}{x^2}}.$$

Using properties of limits, we may evaluate this to be

$$\lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^3}+3}}{1-\frac{3}{x^2}} = \frac{\sqrt{\displaystyle\lim_{x\to\infty}\frac{1}{x^2}+3}}{1-\displaystyle\lim_{x\to\infty}\frac{3}{x^2}} = \frac{\sqrt{0+3}}{1-0} = \sqrt{3}.$$

That is, we have $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x+3x^4}}{x^2-3} = \sqrt{3}$.

(b): Compute the following limit (if it exists), making your steps as clear as possible: $\displaystyle\lim_{x\to-\infty} \frac{x^2-2x^{1/3}+2}{2x^3+2x^{2/3}+1}$.

Solution: As in the previous problem, we look at the exponent of the denominator, which is 3. So we multiply the numerator and the denominator by $1/x^3$. In order to simplify, we have to be careful with the exponents. We have

$$\lim_{x\to-\infty} \frac{x^2-2x^{1/3}+2}{2x^3+2x^{2/3}+1} = \lim_{x\to-\infty} \frac{x^2-2x^{1/3}+2}{2x^3+2x^{2/3}+1}\cdot\frac{1/x^3}{1/x^3} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^3}-\frac{2x^{1/3}}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}+\frac{2x^{2/3}}{x^3}+\frac{1}{x^3}}.$$

Now note that $\frac{2x^{1/3}}{x^3} = \frac{2x^{1/3}}{x^{9/3}} = \frac{2}{x^{8/3}}$ and $\frac{2x^{2/3}}{x^3} = \frac{2x^{2/3}}{x^{9/3}} = \frac{2}{x^{7/3}}$. Using this and simplifying the expression above gives

$$\lim_{x\to-\infty} \frac{\frac{x^2}{x^3}-\frac{2x^{1/3}}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}+\frac{2x^{2/3}}{x^3}+\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{\frac{1}{x}-\frac{2}{x^{8/3}}+\frac{2}{x^3}}{2+\frac{2}{x^{7/3}}+\frac{1}{x^3}}.$$

Finally, we use properties of limits to evaluate:

$$\lim_{x\to-\infty} \frac{\frac{1}{x}-\frac{2}{x^{8/3}}+\frac{2}{x^3}}{2+\frac{2}{x^{7/3}}+\frac{1}{x^3}} = \frac{\displaystyle\lim_{x\to-\infty}\frac{1}{x}-\lim_{x\to-\infty}\frac{2}{x^{8/3}}+\lim_{x\to-\infty}\frac{2}{x^3}}{2+\displaystyle\lim_{x\to-\infty}\frac{2}{x^{7/3}}+\lim_{x\to-\infty}\frac{1}{x^3}} = \frac{0-0+0}{2+0+0} = \frac{0}{2} = 0.$$

That is, we have $\displaystyle\lim_{x\to-\infty} \frac{x^2-2x^{1/3}+2}{2x^3+2x^{2/3}+1} = 0$.