Quiz 3 **Solutions**, Math 111, Section 4 (Vinroot)

(a): Compute the following limit, showing steps clearly:

$$\lim_{x\to 0} \frac{3x}{\sin(4x)}$$

Solution: First, we note that we have $\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{4x} = 1$, since $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, and $x \to 0$ is the same as $4x \to 0$. So if we have a $4x$ instead of a $3x$ in the expression, we can evaluate the limit. We thus multiply the numerator and denominator by 4, and we obtain

$$\lim_{x\to 0} \frac{3x}{\sin(4x)} = \lim_{x\to 0} \frac{4}{4} \cdot \frac{3x}{\sin(4x)} = \lim_{x\to 0} \frac{3}{4} \cdot \frac{4x}{\sin(4x)}.$$

Note that we also have

$$\lim_{x\to 0} \frac{4x}{\sin(4x)} = \lim_{x\to 0} \frac{1}{\frac{\sin(4x)}{4x}} = \frac{1}{\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{4x}} = \frac{1}{1} = 1.$$

Now we finally have

$$\lim_{x\to 0} \frac{3x}{\sin(4x)} = \lim_{x\to 0} \frac{3}{4} \cdot \frac{4x}{\sin(4x)} = \frac{3}{4} \lim_{x\to 0} \frac{4x}{\sin(4x)} = \frac{3}{4} \cdot 1 = \frac{3}{4}.$$

(b): Compute the following derivative, showing steps clearly:

$$\frac{d}{dx}\left(\sin(x^3 + e^x) + \sqrt{e^x + x^2}\right)$$

Solution: First, we have the derivative is the sum of the derivatives of the two functions, and we write the square root as an exponent, so

$$\frac{d}{dx}\left(\sin(x^3 + e^x) + \sqrt{e^x + x^2}\right) = \frac{d}{dx}\left(\sin(x^3 + e^x)\right) + \frac{d}{dx}\left((e^x + x^2)^{1/2}\right).$$

We apply the chain rule to each of these functions, and we obtain

$$\frac{d}{dx}\left(\sin(x^3 + e^x)\right) = \cos(x^3 + e^x)\frac{d}{dx}(x^3 + e^x) = (3x^2 + e^x)\cos(x^3 + e^x),$$

$$\text{and} \quad \frac{d}{dx}\left((e^x + x^2)^{1/2}\right) = \frac{1}{2}(e^x + x^2)^{-1/2}\frac{d}{dx}(e^x + x^2) = \frac{e^x + 2x}{2(e^x + x^2)^{1/2}}.$$

Putting these together, we have the derivative is

$$\frac{d}{dx}\left(\sin(x^3 + e^x) + \sqrt{e^x + x^2}\right) = (3x^2 + e^x)\cos(x^3 + e^x) + \frac{e^x + 2x}{2(e^x + x^2)^{1/2}}.$$