Quiz 1 **Solutions**, Math 111, Section 4 (Vinroot)

(a): Define the function $f(x)$ as follows:

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < 2 \\ 2 + \sqrt{x-2} & \text{if } x > 2. \end{cases}$$

Determine whether $\lim_{x \to 2} f(x)$ exists by calculating one-sided limits, with a brief explanation.

Solution: We must compute $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$, and check if these are equal. Recall that $x \to 2^-$ means that $x < 2$ as x approaches 2, so according to the definition of the function we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 1) = 2^2 - 1 = 3,$$

where we have used algebraic properties of limits in the last step. Since $x \to 2^+$ means $x > 2$, then we have

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2 + \sqrt{x-2}) = 2 + \sqrt{\lim_{x \to 2^+} (x-2)} = 2 + \sqrt{2-2} = 2.$$

Note that since $x > 2$, then the numbers under the square root will be positive as x approaches 2, so that the limit indeed makes sense. Now we have $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, and so $\lim_{x \to 2} f(x)$ **does not exist**.

(b): Compute the following limit if it exists, making your steps clear: $\lim_{t \to 1} \frac{1 - \sqrt{t}}{2 - 2t}$

Solution: If we substitute $t = 1$ in the numerator or denominator, the result is 0, and so we cannot compute the limit in this way. Instead, we algebraically manipulate the expression to look for a cancellation, by factoring the denominator, and rationalizing the numerator by multiplying the top and bottom by $1 + \sqrt{t}$. We obtain

$$\lim_{t \to 1} \frac{1 - \sqrt{t}}{2 - 2t} = \lim_{t \to 1} \frac{1 - \sqrt{t}}{2(1-t)} \cdot \frac{(1 + \sqrt{t})}{(1 + \sqrt{t})} = \lim_{t \to 1} \frac{1 - t}{2(1-t)(1+\sqrt{t})} = \lim_{t \to 1} \frac{1}{2(1 + \sqrt{t})},$$

where we can cancel the factor $(1 - t)$ since $t \neq 1$ as t approaches 1, and so we are not dividing by 0. Now we may use properties of limits to evaluate:

$$\lim_{t \to 1} \frac{1 - \sqrt{t}}{2 - 2t} = \lim_{t \to 1} \frac{1}{2(1 + \sqrt{t})} = \frac{1}{2(1 + \lim_{t \to 1} \sqrt{t})} = \frac{1}{2(1+1)} = \frac{1}{4}.$$