(a): Define the function \( f(x) \) as follows:

\[
f(x) = \begin{cases} 
  x^2 - 1 & \text{if } x < 2 \\
  2 + \sqrt{x-2} & \text{if } x > 2.
\end{cases}
\]

Determine whether \( \lim_{x \to 2} f(x) \) exists by calculating one-sided limits, with a brief explanation.

**Solution:** We must compute \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \), and check if these are equal. Recall that \( x \to 2^- \) means that \( x < 2 \) as \( x \) approaches 2, so according to the definition of the function we have

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 1) = 2^2 - 1 = 3,
\]

where we have used algebraic properties of limits in the last step. Since \( x \to 2^+ \) means \( x > 2 \), then we have

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2 + \sqrt{x-2}) = 2 + \sqrt{2-2} = 2.
\]

Note that since \( x > 2 \), then the numbers under the square root will be positive as \( x \) approaches 2, so that the limit indeed makes sense. Now we have \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \), and so \( \lim_{x \to 2} f(x) \) does not exist.

(b): Compute the following limit if it exists, making your steps clear:

\[
\lim_{t \to 1} \frac{1 - \sqrt{t}}{2 - 2t}
\]

**Solution:** If we substitute \( t = 1 \) in the numerator or denominator, the result is 0, and so we cannot compute the limit in this way. Instead, we algebraically manipulate the expression to look for a cancellation, by factoring the denominator, and rationalizing the numerator by multiplying the top and bottom by \( 1 + \sqrt{t} \). We obtain

\[
\lim_{t \to 1} \frac{1 - \sqrt{t}}{2 - 2t} = \lim_{t \to 1} \frac{1 - \sqrt{t}}{2(1-t)} \cdot \frac{1 + \sqrt{t}}{1 + \sqrt{t}} = \lim_{t \to 1} \frac{1 - t}{2(1-t)(1 + \sqrt{t})} = \lim_{t \to 1} \frac{1}{2(1 + \sqrt{t})},
\]

where we can cancel the factor \( (1-t) \) since \( t \neq 1 \) as \( t \) approaches 1, and so we are not dividing by 0. Now we may use properties of limits to evaluate:

\[
\lim_{t \to 1} \frac{1 - \sqrt{t}}{2 - 2t} = \lim_{t \to 1} \frac{1}{2(1 + \sqrt{t})} = \frac{1}{2(1 + \lim_{t \to 1} \sqrt{t})} = \frac{1}{2(1 + 1)} = \frac{1}{4}.
\]