Quiz 6 **Solution**, Math 111, Section 4 (Vinroot)

(a): Explain why the hypotheses of the Mean-Value Theorem are satisfied, and find all values c which satisfy the Mean-Value Theorem for the function $f(x) = \frac{2}{3}x^3 - x$ on the interval $[-1, 2]$.

Solution: Since $f(x)$ is a polynomial, it is differentiable and continuous at all x, and so f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. We have $f'(x) = 2x^2 - 1$, and the Mean-Value Theorem states that there is some value c in the *open* interval $(-1, 2)$ such that

$$\frac{f(2) - f(-1)}{2 - (-1)} = f'(c) \quad \text{so} \quad \frac{(16/3) - 2 - ((-2/3) + 1)}{3} = 1 = f'(c).$$

That is, we have $2c^2 - 1 = 1$, or $2c^2 = 2$, so $c^2 = 1$. This gives $c = \pm 1$, but the Mean-Value Theorem says that our value c is in the *open* interval $(-1, 2)$, and so the value $c = -1$ must be thrown out. The only value which satisifes the Mean-Value Theorem is $c = 1$.

(b): Suppose f is a function such that $f(1) = 2$ and $f'(x) \leq 2$ for all values of x. What is the largest possible value of $f(4)$ (using the Mean-Value Theorem)?

Solution: Since $f'(x) \leq 2$ for all x, then in particular f is differentiable at all x ($f'(x)$ exists), and so f is continuous at all x. Thus the hypotheses for the Mean-Value Theorem are satisfied for f on any interval. We apply the Mean-Value Theorem to f on the interval $[1, 4]$, since we know $f(1) = 2$ and we want information on $f(4)$. We have that there is some c in the interval $(1, 4)$ such that

$$\frac{f(4) - f(1)}{4 - 1} = f'(c), \quad \text{so} \quad \frac{f(4) - 2}{3} = f'(c) \leq 2,$$

since $f'(x) \leq 2$ for all values of x. Multiplying the last inequality by 3 we have $f(4) - 2 \leq 6$, and so $f(4) \leq 8$. So $f(4)$ can be no larger than 8.