Quiz 5 **Solutions**, Math 111, Section 4 (Vinroot)

(a): Compute y' if $y = (\sec(x))^{\sqrt{x}}$.

Solution: First take the natural logarithm of both sides. Using properties of lograrithms and re-writing $\sqrt{x} = x^{1/2}$ gives

$$\ln(y) = x^{1/2}\ln(\sec(x)).$$

Now we take the derivative of both sides, but we have to remember to use the chain rule on the left side as well. In particular, $\frac{d}{dx}(\ln(y)) = \frac{y'}{y}$. Using the product rule and the chain rule on the right gives:

$$\frac{y'}{y} = \frac{1}{2}x^{-1/2}\ln(\sec(x)) + x^{1/2}\frac{1}{\sec(x)}\frac{d}{dx}(\sec(x)) = \frac{1}{2}x^{-1/2}\ln(\sec(x)) + x^{1/2}\frac{1}{\sec(x)}(\sec(x)\tan(x))$$
$$= \frac{1}{2}x^{-1/2}\ln(\sec(x)) + x^{1/2}\tan(x)$$

Finally, since we need y', we multiply both sides by $y = (\sec(x))^{\sqrt{x}}$. This gives

$$y' = (\sec(x))^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(\sec(x)) + \sqrt{x}\tan(x)\right).$$

(b): The population of bacteria in a dish grows exponentially. It begins with 10 million bacteria cells, and triples after one and a half hours. Find an expression which gives the population at any time, and be sure to specify units.

Solution: Let $P(t)$ be the population in millions of bacteria cells, at time t in hours (you could choose these units differently, then adjust the numbers to fit). We set $P(0) = 10$, so $t = 0$ is the time when the population is 10 million, and then $P(3/2) = 30$ since the popluation triples in 3/2 hours. We know that $P(t) = Ce^{kt}$, where $C = P(0)$, so $P(t) = 10e^{kt}$. We substitute in $t = 3/2$ and $P = 30$ to solve for k. This gives $30 = 10e^{k(3/2)}$. Divide by 10 and then take the natural logarithm, and get:

$$3 = e^{k(3/2)}, \quad \text{and so} \quad \ln(3) = (3/2)k, \quad \text{which gives} \quad k = (2/3)\ln(3).$$

Substituting this back into our equation gives

$$P(t) = 10e^{(2/3)t\ln(3)}.$$

This answer is fine, but we could also write the expoential as $(e^{\ln(3)})^{(2/3)t}$. Since $e^{\ln(3)} = 3$, we would then have

$$P(t) = 10 \cdot 3^{(2/3)t}.$$