Quiz 4 **Solutions**, Math 111, Section 4 (Vinroot)

(a): Compute $f'(x)$ if $f(x) = \sin(3^x) + \ln(\sqrt[3]{x} + x^2)$.

Solution: We use the chain rule for each of the two functions which sum to $f(x)$. We rewrite $\sqrt[3]{x} = x^{1/3}$, and we recall that $\frac{d}{dx}(3^x) = 3^x \ln(3)$ and $\frac{d}{dx}(\ln(x)) = 1/x$. We have

$$\begin{aligned} f'(x) &= \cos(3^x)\frac{d}{dx}(3^x) + \frac{1}{x^{1/3} + x^2}\frac{d}{dx}(x^{1/3} + x^2) \\ &= \cos(3^x)(3^x \ln(3)) + \frac{1}{x^{1/3} + x^2}\left(\frac{1}{3}x^{-2/3} + 2x\right). \end{aligned}$$

That is, we have

$$f'(x) = 3^x \ln(3)\cos(3^x) + \frac{\frac{1}{3}x^{-2/3} + 2x}{x^{1/3} + x^2}.$$

(b): Compute y', given that $\cos(x^2 y) = \sin(xy^2) - ye^x$.

Solution: We use implicit differentiation. That is, we take the derivative with respect to x on both sides, and keep in mind that y is implicitly a function of x so the chain rule must be applied in each case there is a y in an expression. For example, we have $\frac{d}{dx}(y^2) = 2yy'$. Differentiating, we have:

$$\begin{aligned} \frac{d}{dx}(\cos(x^2 y)) &= \frac{d}{dx}(\sin(xy^2) - ye^x) \\ -\sin(x^2 y)(2xy + x^2 y') &= \cos(xy^2)(y^2 + 2yy'x) - y'e^x - ye^x \end{aligned}$$

Now that the derivative is taken, we multiply out each side, and move all summands with a y' term to one side, and all other terms to the other side:

$$\begin{aligned} -2xy\sin(x^2 y) - y'x^2\sin(x^2 y) &= y^2\cos(xy^2) + 2xyy'\cos(xy^2) - y'e^x - ye^x \\ y'e^x - y'x^2\sin(x^2 y) - 2xyy'\cos(xy^2) &= 2xy\sin(x^2 y) + y^2\cos(xy^2) - ye^x \\ y'(e^x - x^2\sin(x^2 y) - 2xy\cos(xy^2)) &= 2xy\sin(x^2 y) + y^2\cos(xy^2) - ye^x \end{aligned}$$

Solving for y' gives

$$y' = \frac{2xy\sin(x^2 y) + y^2\cos(xy^2) - ye^x}{e^x - x^2\sin(x^2 y) - 2xy\cos(xy^2)}.$$