Quiz 1 Solutions, Math 111, Section 5 (Vinroot)

(a): If we define \( f(x) \) as below, compute \( \lim_{x \to 1} f(x) \) if it exists, and explain (using one-sided limits):

\[
f(x) = \begin{cases} 
3x - 2 & \text{if } x < 1 \\
\frac{1}{x} + 2\sqrt{x} - 2 & \text{if } x > 1.
\end{cases}
\]

**Solution:** We have to check if \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \). If they are equal, their common value is equal to \( \lim f(x) \), while if they are not equal (or don’t exist) then the limit in question does not exist. Using the piecewise definition of \( f(x) \), we have (since \( x < 1 \) if \( x \to 1^- \))

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 2) = 3(1) - 2 = 1,
\]

using direct substitution since \( 3x - 2 \) is a polynomial. For the limit from the other side (where \( x > 1 \) since \( x \to 1^+ \)), we have

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left( \frac{1}{x} + 2\sqrt{x} - 2 \right) = \lim_{x \to 1^+} \frac{1}{x} + 2\sqrt{\lim_{x \to 1^+} x} - 2 = \frac{1}{1} + 2(1) - 2 = 1.
\]

Now we can say \( \lim_{x \to 1} f(x) = 1 \) (and exists).

(b): Compute the following limit if it exists, and if it does not exist but is infinite, describe the infinite limit and explain (factor the denominator):

\[
\lim_{u \to 2^+} \frac{u - 3}{u^2 - u - 2}
\]

**Solution:** If we plug in \( u = 2 \) in the denominator, we get 0, but the numerator is not 0. So this must be an infinite limit. Factoring the denominator, we have

\[
\frac{u - 3}{u^2 - u - 2} = \frac{u - 3}{(u - 2)(u + 1)}.
\]

As \( u \to 2^+ \), we have the numerator approaches \( 2 - 3 = -1 < 0 \). In the denominator, we have the factor \( u + 1 \) approaches \( 2 + 1 > 0 \), while \( u - 2 \) approaches 0 but remains positive (since \( u > 2 \) when \( u \to 2^+ \)). So the product \((u - 2)(u + 1)\) remains positive when \( u \to 2^+ \). Since the numerator is negative as \( u \to 2^+ \), then the whole expression is negative. But since the denominator also goes to 0 while the numerator does not, while the expression is negative, the limit must be negatively infinite. That is,

\[
\lim_{u \to 2^+} \frac{u - 3}{u^2 - u - 2} = -\infty.
\]