Quiz 1 **Solutions**, Math 111, Section 5 (Vinroot)

(a): If we define $f(x)$ as below, compute $\lim_{x \to 1} f(x)$ if it exists, and explain (using one-sided limits):

$$f(x) = \begin{cases} 3x - 2 & \text{if } x < 1 \\ \frac{1}{x} + 2\sqrt{x} - 2 & \text{if } x > 1. \end{cases}$$

Solution: We have to check if $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$. If they are equal, their common value is equal to $\lim_{x \to 1} f(x)$, while if they are not equal (or don't exist) then the limit in question does not exist. Using the piecewise definition of $f(x)$, we have (since $x < 1$ if $x \to 1^-$)

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 2) = 3(1) - 2 = 1,$$

using direct substitution since $3x - 2$ is a polynomial. For the limit from the other side (where $x > 1$ since $x \to 1^+$), we have

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(\frac{1}{x} + 2\sqrt{x} - 2 \right) = \lim_{x \to 1^+} \frac{1}{x} + 2\sqrt{\lim_{x \to 1^+} x} - 2 = \frac{1}{1} + 2(1) - 2 = 1.$$

Now we can say $\lim_{x \to 1} f(x) = 1$ (and exists).

(b): Compute the following limit if it exists, and if it does not exist but is infinite, describe the infinite limit and explain (factor the denominator):

$$\lim_{u \to 2^+} \frac{u - 3}{u^2 - u - 2}$$

Solution: If we plug in $u = 2$ in the denominator, we get 0, but the numerator is not 0. So this must be an infinite limit. Factoring the denominator, we have

$$\frac{u - 3}{u^2 - u - 2} = \frac{u - 3}{(u - 2)(u + 1)}.$$

As $u \to 2^+$, we have the numerator approaches $2 - 3 = -1 < 0$. In the denominator, we have the factor $u + 1$ approaches $2 + 1 > 0$, while $u - 2$ approaches 0 but remains positive (since $u > 2$ when $u \to 2^+$). So the product $(u - 2)(u + 1)$ remains positive when $u \to 2^+$. Since the numerator is negative as $u \to 2^+$, then the whole expression is negative. But since the denominator also goes to 0 while the numerator does not, while the expression is negative, the limit must be negatively infinite. That is,

$$\lim_{u \to 2^+} \frac{u - 3}{u^2 - u - 2} = -\infty.$$