Quiz 6 **Solutions**, Math 111, Section 5 (Vinroot)

1. Find all values c which satisfy the Mean-Value Theorem for the function $g(x) = \frac{1}{3}x^3 - 2$ on the interval $[0, 3]$.

Solution: First note that g is continuous and differentiable everywhere, and so the Mean-Value Theorem holds on any interval. So there is some c in the interval $(0, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

where $b = 3$, $a = 0$, and $f'(x) = x^2$. Then $f(3) = \frac{27}{3} - 2 = 9 - 2 = 7$, and $f(0) = 0 - 2 = -2$. So we need c between 0 and 3 such that

$$c^2 = \frac{7 - (-2)}{3 - 0} = 3,$$

which means $c = \sqrt{3}$ or $c = -\sqrt{3}$. Since we need a value between 0 and 3, the only c is $c = \sqrt{3}$.

2. If f is a differentiable function (at all points) such that $f(-1) = 3$ and $f'(x) \leq 1$ for all x, then how large can $f(3)$ be (using the Mean-Value Theorem)?

Solution: Since f is given to be differentiable for all x, then f is also continuous at all x, and the Mean-Value Theorem can be applied to f on the interval $[-1, 3]$. So, there is some c between -1 and 3 such that

$$f'(c) = \frac{f(3) - f(-1)}{3 - (-1)} = \frac{f(3) - 3}{4}.$$

Since $f'(c) \leq 1$, then we have

$$\frac{f(3) - 3}{4} = f'(c) \leq 1, \qquad \text{and so} \qquad f(3) - 3 \leq 4,$$

by multiplying both sides of the inequality by 4. After adding 3 to both sides, we get $f(3) \leq 7$, and so $f(3)$ can be no larger than 7.