Quiz 2 Solutions, Math 111, Section 5 (Vinroot)

Show all steps in the following, and explain if any limits are infinite and/or do not exist.

1. Find the horizontal asymptote(s) of the function \( f(x) = \frac{3x^3 - 2x + 1}{2x^3 + x^2 + 3} \).

Solution: We must compute \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). We divide the numerator and denominator of the expression by \( x^3 \), since 3 is the degree of the denominator. Before taking the limit, we do this and get:

\[
\frac{3x^3 - 2x + 1}{2x^3 + x^2 + 3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{1}{x} + \frac{3}{x^2}}.
\]

Now we may compute the limits as follows:

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left( \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{1}{x} + \frac{3}{x^2}} \right) = \lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{2}{x^2} + \lim_{x \to \infty} \frac{1}{x^3} = \frac{3 - 0 + 0}{2 + 0 + 0} = \frac{3}{2}, \quad \text{and}
\]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} \left( \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{1}{x} + \frac{3}{x^2}} \right) = \lim_{x \to -\infty} 3 - \lim_{x \to -\infty} \frac{2}{x^2} + \lim_{x \to -\infty} \frac{1}{x^3} = \frac{3 - 0 + 0}{2 + 0 + 0} = \frac{3}{2}.
\]

So, the only horizontal asymptote is \( y = \frac{3}{2} \).

2. Compute \( \lim_{x \to \infty} (\sqrt{4x^2 + x} - x) \).

Solution: We rationalize the expression by multiplying the numerator and denominator by \( \sqrt{4x^2 + x} + x \). This gives

\[
\lim_{x \to \infty} (\sqrt{4x^2 + x} - x) \cdot \frac{\sqrt{4x^2 + x} + x}{\sqrt{4x^2 + x} + x} = \lim_{x \to \infty} \frac{4x^2 + x - x^2}{\sqrt{4x^2 + x} + x} = \lim_{x \to \infty} \frac{3x^2 + x}{\sqrt{4x^2 + x} + x}.
\]

Next, the “effective degree” of the denominator is 1, since there is an \( x^2 \) under the square root (which is an exponent of 1/2). So we divide the numerator and denominator by \( x \). Since \( x \to \infty \), then \( x > 0 \), and so \( x = \sqrt{x^2} \). This gives

\[
\lim_{x \to \infty} \frac{3x^2 + x}{\sqrt{4x^2 + x} + x} \cdot \frac{1/x}{1/x} = \lim_{x \to \infty} \frac{3x + 1}{\sqrt{4 + \frac{1}{x}} + 1}.
\]

Now we see, using properties of limits, that the limit of the denominator is

\[
\lim_{x \to \infty} \left( \sqrt{4 + (1/x) + 1} \right) = \sqrt{\lim_{x \to \infty} 4 + \lim_{x \to \infty} (1/x) + \lim_{x \to \infty} 1} = \sqrt{4 + 0 + 1} = 3.
\]

Since the denominator has a nonzero limit, while the numerator has infinite limit,

\[
\lim_{x \to \infty} (3x + 1) = \infty,
\]

then the original limit must also be positively infinite. That is, the limit does not exist, but we have an infinite limit,

\[
\lim_{x \to \infty} (\sqrt{4x^2 + x} - x) = \infty.
\]