Quiz 2 **Solutions**, Math 111, Section 5 (Vinroot)

Show all steps in the following, and explain if any limits are infinite and/or do not exist.

1. Find the horizontal asymptote(s) of the function $f(x) = \dfrac{3x^3 - 2x + 1}{2x^3 + x^2 + 3}$.

Solution: We must compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. We divide the numerator and denominator of the expression by x^3, since 3 is the degree of the denominator. Before taking the limit, we do this and get:

$$\frac{3x^3 - 2x + 1}{2x^3 + x^2 + 3} \cdot \frac{1/x^3}{1/x^3} = \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{1}{x} + \frac{3}{x^3}}.$$

Now we may compute the limits as follows:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{1}{x} + \frac{3}{x^3}}\right) = \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{2}{x^2} + \lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{3}{x^3}} = \frac{3-0+0}{2+0+0} = \frac{3}{2}, \quad \text{and}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{2 + \frac{1}{x} + \frac{3}{x^3}}\right) = \frac{\lim_{x\to-\infty} 3 - \lim_{x\to-\infty}\frac{2}{x^2} + \lim_{x\to-\infty}\frac{1}{x^3}}{\lim_{x\to-\infty} 2 + \lim_{x\to-\infty}\frac{1}{x} + \lim_{x\to-\infty}\frac{3}{x^3}} = \frac{3-0+0}{2+0+0} = \frac{3}{2}.$$

So, the only horizontal asymptote is $y = 3/2$.

2. Compute $\lim_{x\to\infty}(\sqrt{4x^2 + x} - x)$.

Solution: We rationalize the expression by multiplying the numerator and denominator by $\sqrt{4x^2+x} + x$. This gives

$$\lim_{x\to\infty}(\sqrt{4x^2+x} - x) \cdot \frac{\sqrt{4x^2+x}+x}{\sqrt{4x^2+x}+x} = \lim_{x\to\infty}\frac{4x^2 + x - x^2}{\sqrt{4x^2+x}+x} = \lim_{x\to\infty}\frac{3x^2+x}{\sqrt{4x^2+x}+x}.$$

Next, the "effective degree" of the denominator is 1, since there is an x^2 under the square root (which is an exponent of 1/2). So we divide the numerator and denominator by x. Since $x \to \infty$, then $x > 0$, and so $x = \sqrt{x^2}$. This gives

$$\lim_{x\to\infty}\frac{3x^2+x}{\sqrt{4x^2+x}+x} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty}\frac{3x+1}{\sqrt{4+\frac{1}{x}}+1}.$$

Now we see, using properties of limits, that the limit of the denominator is

$$\lim_{x\to\infty}\left(\sqrt{4+(1/x)}+1\right) = \sqrt{\lim_{x\to\infty} 4 + \lim_{x\to\infty}(1/x)} + \lim_{x\to\infty} 1 = \sqrt{4+0}+1 = 3.$$

Since the denominator has a nonzero limit, while the numerator has infinite limit,

$$\lim_{x\to\infty}(3x+1) = \infty,$$

then the original limit must also be positively infinite. That is, the limit does not exist, but we have an infinite limit,

$$\lim_{x\to\infty}(\sqrt{4x^2+x} - x) = \infty.$$