Quiz 1 Solutions, Math 111, Section 2 (Vinroot)

Find values $a$ and $b$ such that the following function is continuous at all real numbers. Give an explanation of your solution, using the definition of continuous.

$$f(x) = \begin{cases} 
  x^2 & \text{if } x < -1 \\
  ax + b & \text{if } -1 \leq x \leq 1 \\
  \sqrt{x + 3} & \text{if } x \geq 1.
\end{cases}$$

Solution: From the definition of $f(x)$, since $x^2$, any linear function $ax + b$, and $\sqrt{x + 3}$ for $x \geq -3$ are all continuous, then $f$ is already continuous for all $x \neq -1, 1$. To force $f$ to be continuous at $x = -1$ and $x = 1$, we need, by the definition of being continuous,

$$\lim_{x \to -1^-} f(x) = f(-1) = -a + b \quad \text{and} \quad \lim_{x \to -1^+} f(x) = f(-1) = -a + b.$$

From the definition of $f$, we have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 = (-1)^2 = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x + 3} = \sqrt{1 + 3} = 2.$$

So, for $f$ to be continuous at $x = -1$ and $x = 1$, we need

$$\lim_{x \to -1^-} f(x) = f(-1) = -a + b \quad \text{and} \quad \lim_{x \to 1^+} f(x) = f(1) = a + b.$$

Therefore, we need $-a + b = 1$ and $a + b = 2$. Adding these together gives $2b = 3$, so $b = 3/2$, and subbing this value of $b$ into either equation gives $a = 1/2$. So the values of $a$ and $b$ must be $a = 1/2$ and $b = 3/2$. 