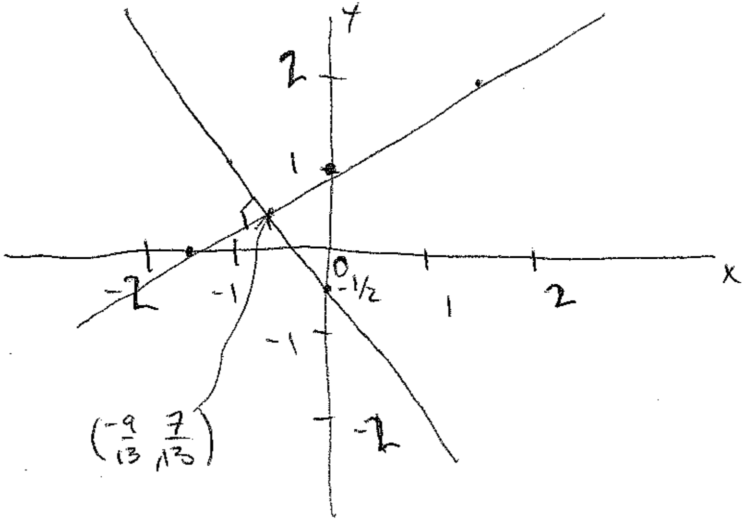
Math 103 - Midterm Review Problems

1. The equation $3y-2x=3$ may be rewritten as $3y=2x+3$, or $y=\frac{2}{3}x+1$, so this line has slope $\frac{2}{3}$. A line perpendicular to it has slope $\frac{-3}{2}$. If this line goes through the point $(-1,1)$, then since its equation is of the form $y=\frac{-3}{2}x+b$, we have $(+1)=\frac{-3}{2}(-1)+b$, so $b=+1+\frac{-3}{2}=\frac{-1}{2}$. The equation is thus $\boxed{y=\frac{-3}{2}x-\frac{1}{2}}$. If these two lines intersect in the point (x,y), then we must have

$$y=\frac{2}{3}x+1=-\frac{3}{2}x-\frac{1}{2}$$ (the y-values for both lines are the same for this x-value)

Since $\frac{2}{3}x+1=\frac{-3}{2}x-\frac{1}{2}$, we have $\frac{2}{3}x+\frac{3}{2}x=-\frac{1}{2}-1$, so $\frac{13}{6}x=-\frac{3}{2}$, so $x=\frac{-3}{\cancel{2}}\cdot\frac{\cancel{6}^{3}}{13}=\frac{-9}{13}$. The corresponding y-value is $y=\frac{2}{\cancel{3}}\left(\frac{-\cancel{9}^{3}}{13}\right)+1=\frac{-6}{13}+1=\frac{7}{13}$. So the point of intersection is $\boxed{\left(\frac{-9}{13},\frac{7}{13}\right)}$. A sketch of these two lines is:



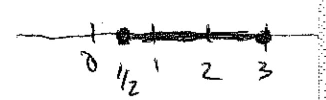
2. $2x^2 - 7x + 3 = (2x-1)(x-3)$, so $(2x-1)(x-3) \le 0$ if

$2x - 1 \le 0$ and $x - 3 \ge 0$, or, $2x - 1 \ge 0$ and $x - 3 \le 0$,

so $\underbrace{x \le \frac{1}{2} \text{ and } x \ge 3}_{\text{not possible}}$, or $x \ge \frac{1}{2}$ and $x \le 3$,

so $x \ge \frac{1}{2}$ and $x \le 3$, which

means $\boxed{\frac{1}{2} \le x \le 3}$.

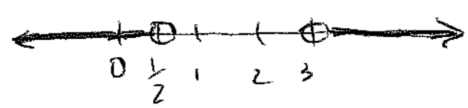


Since $2x^2 - 7x + 3 \le 0$ when $\frac{1}{2} \le x \le 3$, and

$2x^2 - 7x + 3$ must be either ≤ 0, or > 0, then we

must have $2x^2 - 7x + 3 > 0$ for all other x,

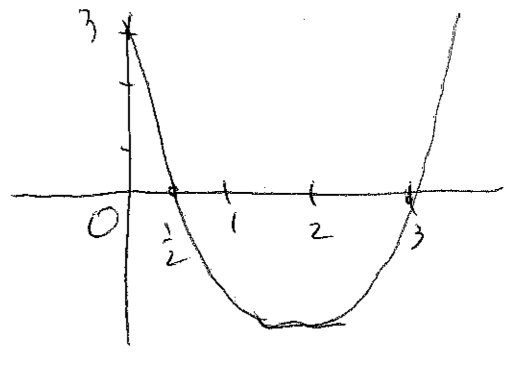
so when $\boxed{x < \frac{1}{2} \text{ or } x > 3}$



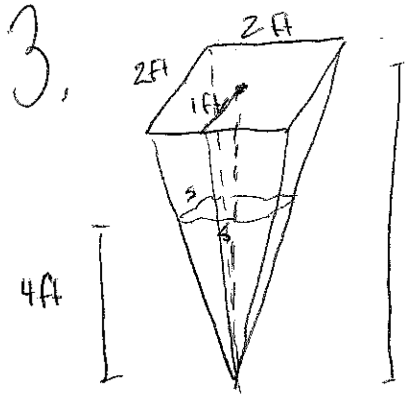
If $y = 2x^2 - 7x + 3$, then we can use this to see

when $y \le 0$ and

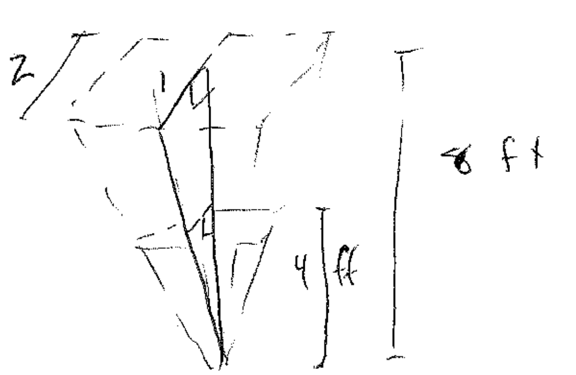
when $y > 0$:

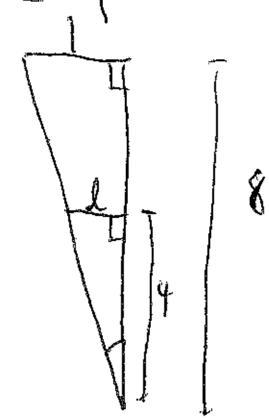


3.



The container is as pictured. The volume of the liquid is the volume of the pyramid of height 4 ft. If the area of its base is B, the volume is $V = \frac{1}{3}Bh$, where $h = 4$ ft, so is $\frac{1}{3}B(4) = \frac{4}{3}B$ ft^3. The base of this smaller pyramid is also square, and if its side length is s, then $B = s^2$. To find s, we use similar triangles. Draw a height of the container from the center of the base to the point (bottom) of the pyramid. This forms a triangle pictured below:





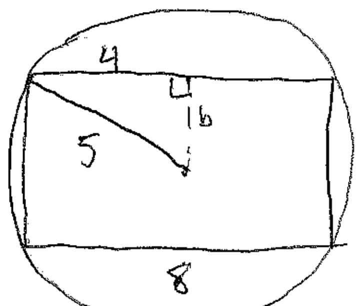
The base of the large triangle (on top) is 1 ft since it is half of the side length of the base of the pyramid. The height is 8 ft. The water forms a smaller similar triangle with height 4, and base $l = \frac{1}{2}s$. Since these are similar triangles, we have $\frac{1}{8} = \frac{l}{4}$, so $l = \frac{4}{8} = \frac{1}{2}$.

3. (cont'd): Since $\frac{1}{2}s = \ell = \frac{1}{2}$, then $s = 1$ ft.

Now the pyramid of water has volume

$$V = \frac{1}{3}Bh = \frac{1}{3}s^2h = \frac{1}{3}(1\text{ ft})^2(4\text{ ft}) = \boxed{\frac{4}{3}\text{ ft}^3}$$

4.



An important observation is that a radius of the circle can be drawn so that it is half of a diagonal of the rectangle, as pictured. A right triangle with hypotenuse 5 and one side 4 is formed (4 being half of the edge of the rectangle of length 8). The other side, say b, can be found using the Pythagorean Theorem: $b^2 + 4^2 = 5^2$, so $b^2 = 25 - 16 = 9$, so $b = 3$. This is half of the other side length of the rectangle, so the other side has length $2b = \boxed{6}$. The area of the rectangle is then $A = 6 \cdot 8 = \boxed{48}$ and its perimeter is $6 + 6 + 8 + 8 = 28$. The circumference of the circle is $2\pi r = 2\pi \cdot 5 = 10\pi$, and the ratio of the perimeter to the circumference is $\frac{28}{10\pi} = \boxed{\frac{14}{5\pi}}$, or $\boxed{14 : 5\pi}$.

5. (a): $3x^2 - 17x + 10 = (3x-2)(x-5)$, so

$3x^2 - 17x + 10 = 0$ when $3x - 2 = 0$ or $x - 5 = 0$,

so $\boxed{x = \frac{2}{3} \text{ or } x = 5}$

(b): $x^2 + 4x + 1 = 0$. To complete the square we need

a term $\left(\frac{4}{2}\right)^2 = 4$, so we write

$x^2 + 4x + 1 = x^2 + 4x + 4 - 4 + 1 = (x^2 + 4x + 4) - 3$

$= (x+2)^2 - 3 = 0$. Now we can solve:

$(x+2)^2 = 3$, so $x + 2 = \pm\sqrt{3}$, so $x = -2 \pm \sqrt{3}$. So

$\boxed{x = -2 + \sqrt{3} \text{ or } -2 - \sqrt{3}}$.

(c): We cannot easily factor, so we use the quadratic formula for $x^2 + 2x + 6 = 0$, with $a = 1$, $b = 2$, $c = 6$.

Now $b^2 - 4ac = 2^2 - 4(1)(6) = -20 < 0$, so there

are $\boxed{\text{no real solutions}}$.

(d) We first try synthetic division/substitution

for $x^3 + x^2 - 7x + 5 = 0$, and divisors of 5, to

try as solutions, are $\pm 1, \pm 5$. We see 1 works:

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -7 & 5 \\ & & 1 & 2 & -5 \\ \hline & 1 & 2 & -5, & \boxed{0} \end{array}$$

remainder 0

quotient $x^2 + 2x - 5$.

5. (d) (cont'd) The synthetic substitution gives us the factorization

$$x^3+x^2-7x+5 = (x-1)(x^2+2x-5) = 0,$$ so

$x-1=0$ or $x^2+2x-5=0$. The solutions are $x=1$, and the solutions to $x^2+2x-5=0$. We can't easily factor this quadratic, so we use the quadratic formula with $a=1, b=2, c=-5$, so

$b^2-4ac = 2^2-4(1)(-5) = 24 \geq 0$, so there are solutions.

These solutions are $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-2 \pm \sqrt{24}}{2} = \dfrac{-2 \pm 2\sqrt{6}}{2} =$

$= -1 \pm \sqrt{6}$. The solutions to $x^3+x^2-7x+5=0$ are now found to be $\boxed{x = 1, -1+\sqrt{6}, \text{ or } -1-\sqrt{6}}$.

(e): To get rid of the logarithms in

$$\log_3(x) + \log_3(x+2) = 1, \text{ or } \log_3(x(x+2)) = 1,$$

we raise 3 to the power given by each side of the equation. So $3^{\log_3(x(x+2))} = 3^1$, and since $b^{\log_b y} = y$, we have $3^{\log_3(x(x+2))} = x(x+2)$.

So $x(x+2) = 3$, or $x^2+2x=3$, or $x^2+2x-3=0$.

5. (e) (cont'd): Factoring, $(x+3)(x-1)=0$, so $x=-3$ or $x=1$. However, with $x=-3$ in the original equation, $\log_3(-3)$ (and $\log_3(-1)$) is undefined ($\log_b x$ is only defined if $x>0$). So $x=-3$ is not a solution. But $x=1$ does work, and $\boxed{x=1}$ is the only solution.

(f): $\frac{x^2-3x+7}{x^2+2x-6} = 1$, and multiplying both sides by x^2+2x-6 gives $x^2-3x+7 = x^2+2x-6$, so $-3x+7 = 2x-6$, so $5x=13$, or $\boxed{x=\frac{13}{5}}$.

(Note that this does not make the denominator 0, so the original expression really is 1, and not undefined).

(g): We can first write both sides of the equation as powers of 3, where $\frac{1}{9}3^{2x^2} = 3^{-2}\cdot 3^{2x^2} = 3^{2x^2-2}$ and $27\cdot 3^{1/9} = 3^3\cdot 3^{1/9} = 3^{28/9}$. So $3^{2x^2-2} = 3^{28/9}$, and taking $\log_3$ of both sides gives $\log_3\left(3^{2x^2-2}\right) = \log_3\left(3^{28/9}\right)$. Since $\log_b(b^x)=x$, 5.(g) (cont'd) we have $2x^2 - 2 = \frac{28}{9}$, so $2x^2 = 2 + \frac{28}{9} = \frac{46}{9}$.

So $x^2 = \frac{1}{2} \cdot \frac{46}{9} = \frac{23}{9}$, and $x = \pm\sqrt{\frac{23}{9}} = \boxed{\frac{\pm\sqrt{23}}{3}}$.

(h): $A^2 x^{3/2} - P \ln(A^3 B) = My^3$, so

$$A^2 x^{3/2} = P \ln(A^3 B) + My^3, \text{ so}$$

$$x^{3/2} = \frac{1}{A^2}\left(P \ln(A^3 B) + My^3\right).$$

To solve for x, now raise both sides to the $\frac{2}{3}$ power, since $(x^{3/2})^{2/3} = x^{\frac{3}{2}\cdot\frac{2}{3}} = x^1 = x$:

$$\boxed{x = \left[\frac{1}{A^2}\left(P \ln(A^3 B) + My^3\right)\right]^{2/3}}$$

6. (a) We know that $\log_b y$ is only defined if $y > 0$, and is undefined if $y \leq 0$. So $\log_5(7x-9)$ is undefined exactly when $7x - 9 \leq 0$, so when $\boxed{x \leq \frac{9}{7}}$.

(b) The rational expression $\frac{14x^4 - 13}{x^3 + 7x^2 + 15x + 9}$ is undefined exactly when the denominator is 0, so when $x^3 + 7x^2 + 15x + 9 = 0$. Like in 5(d), we use synthetic substitution/division with the divisors of 9,

6. (b) (cont'd): which are $\pm 1, \pm 3, \pm 9$. Note that positive numbers won't work because if we plug a positive number in $x^3+7x^2+15x+9$, the result is positive, and so cannot be 0. If we try -1, though, we find it works:

$$\begin{array}{r|rrrr} -1 & 1 & 7 & 15 & 9 \\ & & -1 & -9 & -9 \\ \hline & 1 & 6 & 9 & \boxed{0} \end{array}$$

remainder $= 0$

quotient is x^2+6x+9.

This gives us the factorization

$$x^3+7x^2+15x+9 = (x+1)(x^2+6x+9).$$

Also $x^2+6x+9 = (x+3)^2$, so $x^3+7x^2+15x+9=0$ when $(x+1)(x+3)^2 = 0$, so when $\boxed{x=-1 \text{ or } -3}$.

(c): We know that $y^{1/n} = \sqrt[n]{y}$, and when n is even, then this is only defined if $y \geq 0$.

So $(-2x^2+x+3)^{1/10} = \sqrt[10]{-2x^2+x+3}$ is undefined when $-2x^2+x+3<0$, or when

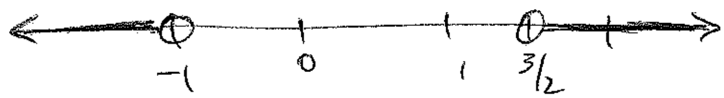
$(-2x+3)(x+1) < 0$, so when

$-2x+3<0$ and $x+1>0$, or, $-2x+3>0$ and $x+1<0$,

6. (c): so $3<2x$ and $x>-1$, or $3>2x$ and $x<-1$,

so $x>\frac{3}{2}$ and $x>-1$, or $x<\frac{3}{2}$ and $x<-1$,

so $\boxed{x>\frac{3}{2} \quad \text{or} \quad x<-1}$.



(d): We also know that when n is odd, $y^{1/n}=\sqrt[n]{y}$ is always defined for all y. So, since 7 is odd, $(x^7-9x^6+x^3-2x+12)^{1/7}$ is defined no matter what value $x^7-9x^4 * x^3-2x+12$ takes, and so for any value of x. So the expression is $\boxed{\text{never undefined}}$.

(e) In the expression $\frac{1}{\ln(x^2+2x-3)}$, we need the denominator to be nonzero for the expression to be defined, and we need the logarithm to be taken of a positive number, so also $x^2+2x-3>0$ for this to be defined. In other words, the expression is undefined if either $x^2+2x-3\leq 0$ or if $\ln(x^2+2x-3)=0$. First, $x^2+2x-3=(x+3)(x-1)\leq 0$

6. (e) (cont'd) if $x+3 \le 0$ and $x-1 \ge 0$, or, $x+3 \ge 0$ and $x-1 \le 0$,

so $\underbrace{x \le -3 \text{ and } x \ge 1}_{\text{(impossible)}}$, or, $x \ge -3$ and $x \le 1$

So $x \ge -3$ and $x \le 1$,

so $-3 \le x \le 1$.

For the other possibility, $\ln(x^2+2x-3) = 0$ when $e^0 = x^2+2x-3$, or $1 = x^2+2x-3$, so $x^2+2x-4=0$. We cannot factor easily, so we use the quadratic formula with $a=1$, $b=2$, $c=-4$, so $b^2-4ac = 4-4(1)(-4) = 20 \ge 0$, so there are solutions. The solutions are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}.$$

Putting all of this together, the expression is undefined if $\boxed{-3 \le x \le 1 \text{ or if } x = -1+\sqrt{5} \text{ or } -1-\sqrt{5}}$

[We could further note that ~~sense~~ since $2 < \sqrt{5} < 3$, then $-1+\sqrt{5}$ satisfies: $1 < -1+\sqrt{5} < 2$, and $-1-\sqrt{5}$ satisfies $-4 < -1-\sqrt{5} < -3$, so the last two values are not in the interval $-3 \le x \le 1$]

6. (f) Since $\ln y$ is undefined when $y \leq 0$, then $\ln(x+4) + \ln(x-2)$ is undefined when

$x+4 \leq 0$ or $x-2 \leq 0$, so when

$x \leq -4$ or $x \leq 2$, so when $\boxed{x \leq 2}$

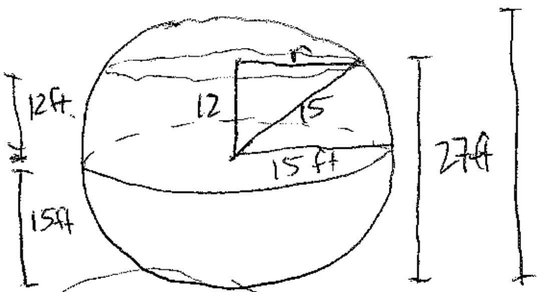
If we rewrote the expression as $\ln((x+4)(x-2))$, then we might say the original expression is undefined when $(x+4)(x-2) < 0$, so when

$x+4 \leq 0$ and $x-2 \geq 0$, or, $x+4 \geq 0$ and $x-2 \leq 0$.

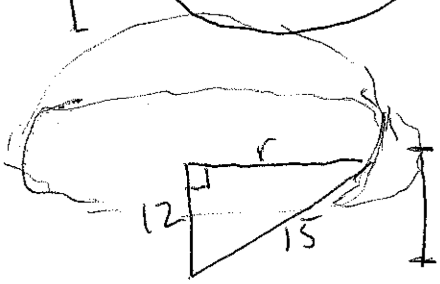
The problem is that this is too restrictive, since we really only need <u>one</u> of $\ln(x+4)$ or $\ln(x-2)$ to be undefined, <u>or</u> both of them. So $x+4 \leq 0$ and $x-2 \leq 0$ is also possible (when $\ln(x+4)$ <u>and</u> $\ln(x-2)$ are undefined), which is left out if we consider $(x+4)(x-2) \leq 0$.

That is, when $\ln(x+4)$ and $\ln(x-2)$ are undefined, we have $(x+4)(x-2) \geq 0$.

7. The water tank is as pictured:



Since the water is at a height of 27 ft, and the radius is 15 ft, the water is at a height of 12 ft above the center of the sphere.



The key observation is that a radius of 15 ft is also given by the segment from the center of the sphere to the edge of the water's surface. If r is the radius of the surface of the water, then we have a right triangle as pictured above. By the Pythagorean theorem, $15^2 = 12^2 + r^2$, so $225 = 144 + r^2$, so $r^2 = 81$, and $\boxed{r = 9 \text{ ft}}$.

8. (a): The long division is as follows:

$$
\begin{array}{r l}
 & x^3 - x + 3 \leftarrow \text{quotient} \\
x^2+1 \,\big) & x^5 + 0x^4 + 0x^3 + 3x^2 + 0x + 2 \\
 & -(x^5 \quad + x^3) \\
 & \quad -x^3 + 3x^2 + 0x + 2 \\
 & \quad -(-x^3 \quad - x) \\
 & \qquad 3x^2 + x + 2 \\
 & \qquad -(3x^2 \quad + 3) \\
 & \qquad\quad x - 1 \leftarrow \text{remainder}
\end{array}
$$

The quotient is $x^3 - x + 3$ and the remainder is $x - 1$. So

$$\frac{x^5 + 3x^2 + 2}{x^2 + 1} = \boxed{x^3 - x + 3 + \frac{x-1}{x^2+1}}$$

8 (b): The long division is as follows:

$$\begin{array}{r|l} & 5x + 4 \\ \hline x^3+x+1 & 5x^4+4x^3+3x^2+2x+1 \\ & -(5x^4 \quad\; +5x^2+5x) \\ \hline & \quad 4x^3 -2x^2-3x+1 \\ & \quad -(4x^3 \quad\; +4x+4) \\ \hline & \qquad -2x^2-7x-3 \end{array}$$

The quotient is $5x+4$, and the remainder is $-2x^2-7x-3$.

So $$\frac{5x^4+4x^3+3x^2+2x+1}{x^3+x+1} = \boxed{5x+4 + \frac{-2x^2-7x-3}{x^3+x+1}}$$

9. (a):

$$2\ln(x^2) + \frac{1}{3}\ln(x) - \frac{\log_2(x^{1/2})}{\log_2 e}$$

Since $\frac{\log_b y}{\log_b x} = \log_x y$, $\frac{\log_2(x^{1/2})}{\log_2 e} = \log_e(x^{1/2}) = \ln(x^{1/2})$

$$= 2\ln(x^2) + \frac{1}{3}\ln(x) - \ln(x^{1/2})$$

$$= 2\cdot 2\ln(x) + \frac{1}{3}\ln(x) - \frac{1}{2}\ln(x)$$

since $\ln(x^n) = n\ln(x)$

$$= 4\ln(x) + \frac{1}{3}\ln(x) - \frac{1}{2}\ln(x)$$

$$= \left(4+\frac{1}{3}-\frac{1}{2}\right)\ln(x) = \left(\frac{24}{6}+\frac{2}{6}-\frac{3}{6}\right)\ln(x)$$

$$= \boxed{\frac{23}{6}\ln(x)}$$

9.(b): $\left(8^{-2/3}\, 2^3 \left(\frac{1}{16}\right)^{-4} 32^{3/5}\right)^2$, $\quad 8=2^3,\ \frac{1}{16}=2^{-5},\ 32=25$

$= \left((2^3)^{-2/3}\, 2^3 \left(2^{-4}\right)^{-4} (2^5)^{3/5}\right)^2$ $\quad (2^a)^b = 2^{ab}$

$= \left(2^{-2}\, 2^3\, 2^{16}\, 2^3\right)^2$ $\quad 2^c 2^d = 2^{c+d}$

$= \left(2^{-2+3+16+3}\right)^2$

$= (2^{20})^2 = \boxed{2^{40}}$

10. (a): $\log_3\left(\frac{1}{9}\right) = \boxed{-2}$ since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(b): $16^{3/4} = \boxed{8}$ since $16^{3/4} = (16^{1/4})^3 = (\sqrt[4]{16})^3 = 2^3 = 8$.

(c) $47^{\log_{47} 4} = \boxed{4}$ since $x = b^{\log_b x}$

(d) $4^{1/4}\, 8^{-3/2} = (2^2)^{1/4} (2^3)^{-3/2} = 2^{1/2}\, 2^{-9/2} = 2^{-8/2} = 2^{-4} = \boxed{\frac{1}{16}}$

(e) $\log_3\left(28^{8 \ln 1}\right)$: The key is $\ln 1 = 0$ since $e^0 = 1$.

So $\log_3\left(28^{8\ln 1}\right) = \log_3\left(28^{8\cdot 0}\right) = \log_3\left(28^0\right) = \log_3 1$

$= \boxed{0}$ since $3^0 = 1$.

(f) $8^{\log_2(3)} = (2^3)^{\log_2 3} = 2^{3\log_2 3} = 2^{\log_2(3^3)} = 2^{\log_2(27)} = \boxed{27}$

(g) $\dfrac{\log_{11} \sqrt[5]{49}}{\log_{11} 7} = \log_7 \sqrt[5]{49} = \log_7 \sqrt[5]{7^2} = \log_7\left(7^{2/5}\right) = \boxed{\frac{2}{5}}$

#10. (h): $\log_{1/10}(.001) = \log_{1/10}(10^{-3}) = \log_{1/10}\left(\frac{1}{10^3}\right) =$

$= \log_{1/10}\left(\left(\frac{1}{10}\right)^3\right) = \boxed{3}.$

(i): $\frac{\ln(17/5)}{\ln 3} \; \frac{2\log_7(\sqrt{3})}{\log_7(3.4)}$.

First note that $2\log_7(\sqrt{3}) = \log_7(\sqrt{3})^2 = \log_7 3$,

and $3.4 = 17/5$, so $\log_7(3.4) = \log_7(17/5)$.

Since $\frac{\log_b y}{\log_b x} = \log_x y$, we have $\frac{\log_7 3}{\log_7(17/5)} = \log_{(17/5)} 3 =$

$= \frac{\log_e 3}{\log_e(17/5)} = \frac{\ln 3}{\ln(17/5)}$.

Now $\frac{\ln(17/5)}{\ln(3)} \; \frac{2\log_7(\sqrt{3})}{\log_7(3.4)} = \frac{\cancel{\ln(17/5)}}{\cancel{\ln(3)}} \; \frac{\cancel{\ln(3)}}{\cancel{\ln(17/5)}} = \boxed{1}.$