Math 103 - HW #6A, B

A1. (a) $240° = \overset{120\cdot 2}{\cancel{240}°} \cdot \dfrac{2\pi}{\underset{120\cdot 3}{\cancel{360}°}} = \boxed{\dfrac{4\pi}{3} \text{ radians}}$ $\quad \dfrac{{}^{2}\cancel{4\pi}/3}{2\cancel{\pi}} = \boxed{\dfrac{2}{3} \text{ of a circle}}$

(b) $450° = 360° + 90° = 2\pi + \dfrac{\pi}{2} = \boxed{\dfrac{5\pi}{2} \text{ radians}}$ $\quad \boxed{1\tfrac{1}{4} = \tfrac{5}{4} \text{ of a circle}}$

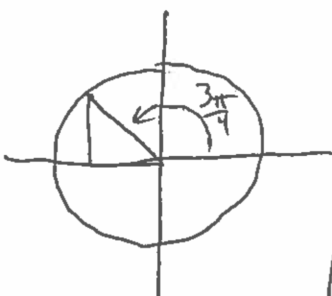
(c) $72° = \overset{12\cdot 6}{\cancel{72}°} \cdot \dfrac{2\pi}{\underset{12\cdot 30\ 5}{360°}} = \boxed{\dfrac{2\pi}{5} \text{ radians}}$ $\quad \boxed{\dfrac{1}{5} \text{ of a circle}}$

(d) $\dfrac{7\pi}{4} = \dfrac{7\cancel{\pi}}{\cancel{4}} \cdot \dfrac{\overset{90\ 45}{\cancel{360}°}}{2\cancel{\pi}} = \boxed{315°}$ $\quad \dfrac{7\cancel{\pi}/4}{2\cancel{\pi}} = \boxed{\dfrac{7}{8} \text{ of a circle}}$

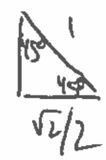
(e) $\dfrac{6\pi}{5} = \dfrac{\overset{3}{\cancel{6}}\cancel{\pi}}{\cancel{5}} \cdot \dfrac{\overset{72}{\cancel{360}°}}{2\cancel{\pi}} = \boxed{216°}$ $\quad \dfrac{{}^{3}\cancel{6}\cancel{\pi}/5}{\cancel{2}\cancel{\pi}} = \boxed{\dfrac{3}{5} \text{ of a circle}}$

(f) $\dfrac{11\pi}{8} = \dfrac{11\cancel{\pi}}{\cancel{8}_2} \cdot \dfrac{\overset{\cancel{180}\ 45}{\cancel{360}°}}{2\cancel{\pi}} = \boxed{247.5°}$ $\quad \dfrac{11\cancel{\pi}/8}{2\cancel{\pi}} = \boxed{\dfrac{11}{16} \text{ of a circle}}$

A2. (a)

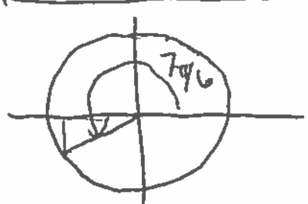


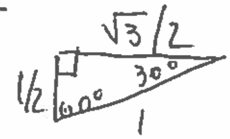
$\dfrac{3\pi}{4} = \dfrac{\pi}{2} + \dfrac{\pi}{4} = \pi - \dfrac{\pi}{4}$



$$\boxed{\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}}$$

(b) $\dfrac{7\pi}{6} = \pi + \dfrac{\pi}{6}$





$$\boxed{\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}}$$

A2 (c) $40\pi = 20 \cdot 2\pi$, so $\sin(40\pi) = \sin(0)$

$= 0 + 2\pi(20)$ since $\sin(\theta + 2\pi k) = \sin(\theta)$ when k is any integer. ($k = 20$ here)

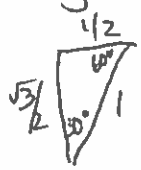
So $\boxed{\sin(40\pi) = \sin(0) = 0}$

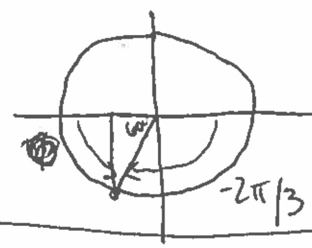
(d) $-\frac{8\pi}{3} = -\frac{6\pi}{3} - \frac{2\pi}{3} = -2\pi - \frac{2\pi}{3}$

So $\cos\left(-\frac{8\pi}{3}\right) = \cos\left(-2\pi - \frac{2\pi}{3}\right) = \cos\left(-\frac{2\pi}{3}\right)$

since $\cos(\theta + 2\pi k) = \cos(\theta)$ when k is any integer. ($k = -1$ here)

$-\frac{2\pi}{3} = -\frac{3\pi}{3} + \frac{\pi}{3} = -\pi + \frac{\pi}{3}$





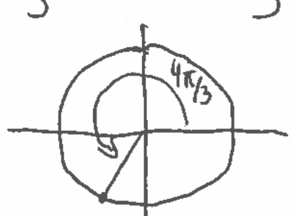
$$\boxed{\cos\left(\frac{-8\pi}{3}\right) = \cos\left(\frac{-2\pi}{3}\right) = -\frac{1}{2}}$$

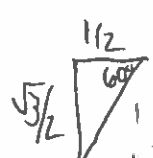
Or: $-\frac{8\pi}{3} = -\frac{12\pi}{3} + \frac{4\pi}{3} = -4\pi + \frac{4\pi}{3}$.

So $\cos\left(\frac{-8\pi}{3}\right) = \cos\left(\frac{4\pi}{3} + -2(2\pi)\right) = \cos\left(\frac{4\pi}{3}\right)$

since $\cos(\theta + 2\pi k) = \cos(\theta)$ for any integer k, ($k = -2$ here).

Now $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

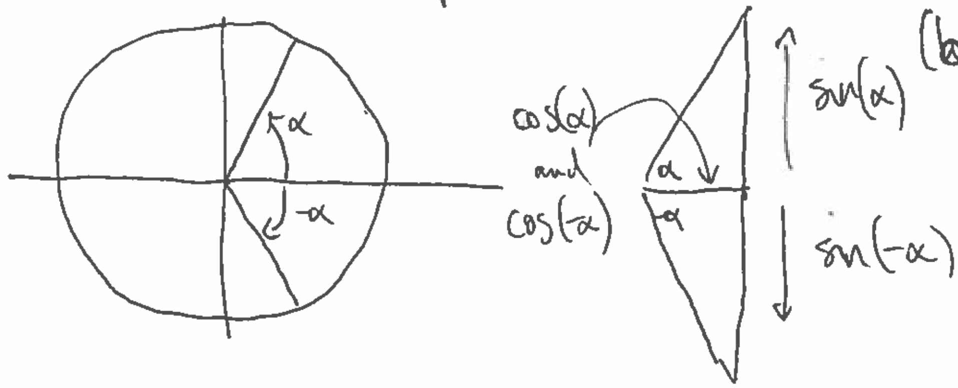




So $\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$,

so $\boxed{\cos\left(\frac{-8\pi}{3}\right) = -\frac{1}{2}}$

A3. General picture:



(b) From the picture, $\sin(-\alpha)$ is the same <u>magnitude</u>, or absolute value as $\sin(\alpha)$, but in the opposite direction in terms of the coordinate.

So $\sin(-\alpha) = -\sin(\alpha)$.

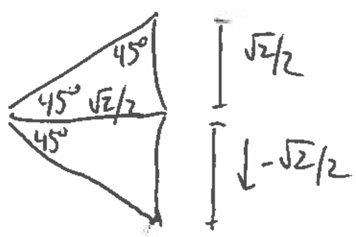
(a) Also in the picture, we see that $\cos(\alpha)$ and $\cos(-\alpha)$ is the exact same coordinate, since it is the exact same side of the corresponding triangle.

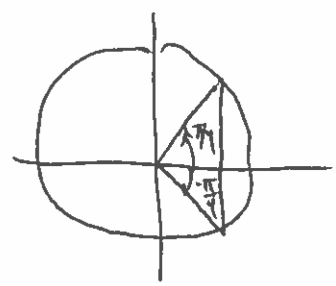
So $\cos(-\alpha) = \cos(\alpha)$

(I switched the order of (a) and (b) above, sorry!)

If $\alpha = \frac{\pi}{4}$, the picture looks like:

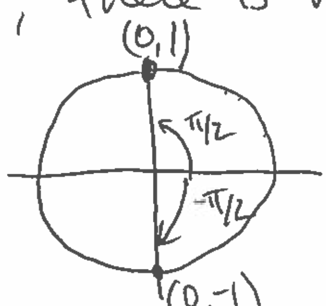
$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{-\pi}{4}\right) = \frac{\sqrt{2}}{2}$$





$$\sin\left(\frac{-\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

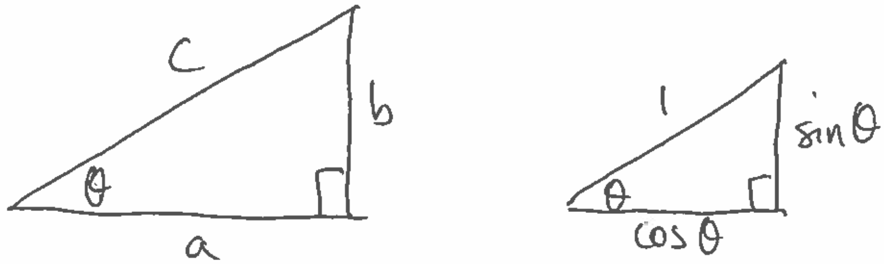
If $\alpha = \frac{\pi}{2}$, there is no triangle, but the statement still holds:



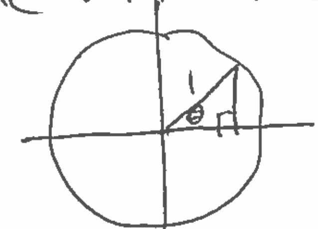
$$\sin\left(\frac{-\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$$

$$\cos\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0.$$

A4. Here is a general right triangle:



It is similar to a right triangle with the same angles, but with hypotenuse 1.



From the unit circle meaning of $\cos\theta$ and $\sin\theta$, the other two sides of this triangle are exactly $\cos\theta$ and $\sin\theta$.

Since the triangles are similar, we have proportions:

$$\frac{c}{1} = \frac{b}{\sin\theta}, \text{ so } \boxed{b = c\sin\theta} \text{ or } \sin\theta = \frac{b}{c}, \text{ and}$$

$$\frac{c}{1} = \frac{a}{\cos\theta}, \text{ so } \boxed{a = c\cos\theta} \text{ or } \cos\theta = \frac{a}{c}.$$

(This is where $\cos\theta$ = "adjacent over hypotenuse" and $\sin\theta$ = "opposite over hypotenuse" come from)

If $c = 3$ and $\theta = \frac{\pi}{6}$, then $a = 3\cos\left(\frac{\pi}{6}\right) = \boxed{\frac{3\sqrt{3}}{2}}$

and $b = 3\sin\left(\frac{\pi}{6}\right) = \boxed{\frac{3}{2}}$

If $c = 2$ and $\theta = \frac{\pi}{4}$, then $a = 2\cos\left(\frac{\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2} = \boxed{\sqrt{2}}$

and $b = 2\sin\left(\frac{\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2} = \boxed{\sqrt{2}}$.

A5. We first get all terms with y in them on one side:

$$3yx + 4 = 4xy + (2y + 3x)\cos(2xt)$$

$$3yx + 4 = 4xy + 2y\cos(2xt) + 3x\cos(2xt)$$

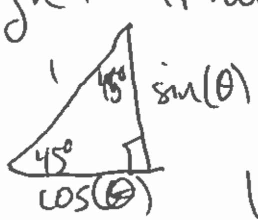
So $4 - 3x\cos(2xt) = xy + 2y\cos(2xt)$.

Now factor out a y on the left side and divide by the rest: $4 - 3x\cos(2xt) = y(x + 2\cos(2xt))$

So $\boxed{y = \dfrac{4 - 3x\cos(2xt)}{x + 2\cos(2xt)}}$ (This cannot be further simplified).

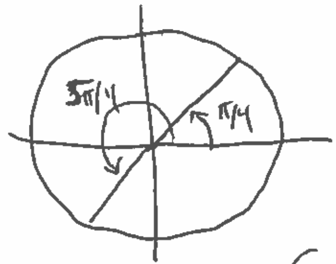
A6. The expression $\dfrac{x^3 - 17x^2 - 3x + 2}{\sin(x) - \cos(x)}$ is undefined when the denominator is 0, so when $\sin(x) - \cos(x) = 0$, so when $\sin(x) = \cos(x)$.

For an angle θ, $\sin(\theta) = \cos(\theta)$ when the corresponding right triangle on the unit circle is isoceles;



so on a 45°-45° right triangle.

When restricting angles between 0 and 2π (so only once around the unit circle), A6 (cont'd) this happens exactly twice:



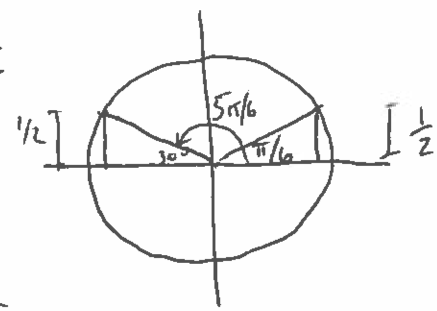
For angles $\frac{\pi}{4}$, when $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

and $\frac{5\pi}{4}$, when $\cos\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

So for $0 \le x < 2\pi$, the only values making the expression undefined are

$$\boxed{x = \frac{\pi}{4} \text{ or } x = \frac{5\pi}{4}}$$

A7. $2\sin(x) = 1$ exactly when $\sin(x) = \frac{1}{2}$ (by dividing both sides by 2). If we just look at angles between 0 and 2π (once around the unit circle), then $\sin(x) = \frac{1}{2}$ exactly two times:



For angles $\frac{\pi}{6}$ and $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$

So, $\sin(x) = \frac{1}{2}$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ for $0 \le x \le 2\pi$.

However, $\sin(x + 2\pi k) = \sin(x)$ for any integer k.

So all solutions are given by

$$\boxed{x = \frac{\pi}{6} + 2\pi k \text{ or } \frac{5\pi}{6} + 2\pi k \text{ for } \underline{\text{any}} \text{ integer } k.}$$

A8. Since $\cos^2 x + \sin^2 x = 1$ for any x, then $\cos^2 x = 1 - \sin^2 x$. In the equation $-2\cos^2 x - 7\sin x + 5 = 0$, substitute $\cos^2 x = 1 - \sin^2 x$ to obtain

$$-2(1-\sin^2(x)) - 7\sin(x) + 5$$
$$= -2 + 2\sin^2(x) - 7\sin(x) + 5$$
$$= 2\sin^2(x) - 7\sin(x) + 3 = 0.$$

This is a quadratic in $\sin(x)$ which we can factor. That is, if we temporarily let $w = \sin(x)$, we have $2w^2 - 7w + 3 = (2w-1)(w-3) = 0$, so

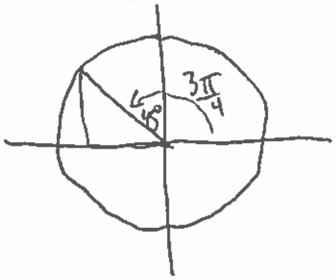
$$(2\sin(x) - 1)(\sin(x) - 3) = 0.$$

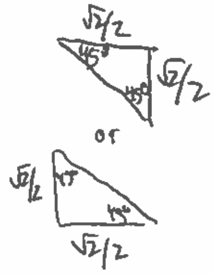
So $2\sin(x) - 1 = 0$, so $2\sin(x) = 1$, or
$\sin(x) - 3 = 0$, so $\sin(x) = 3$.

For any x, from the unit circle meaning of $\sin(x)$, we know $-1 \leq \sin(x) \leq 1$. So $\sin(x) = 3$ is impossible and has no solutions. The solutions to $2\sin(x) = 1$, or $\sin(x) = \frac{1}{2}$, were worked out in A7, and the solutions are

$$\boxed{x = \frac{\pi}{6} + 2\pi k \text{ or } \frac{5\pi}{6} + 2\pi k \text{ for any integer } k}$$

B1. (a) $\sec\left(\frac{3\pi}{4}\right) = \frac{1}{\cos\left(\frac{3\pi}{4}\right)}$, $\frac{3\pi}{4} = \frac{\pi}{2} + \frac{\pi}{4}$



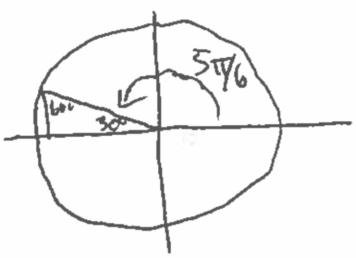


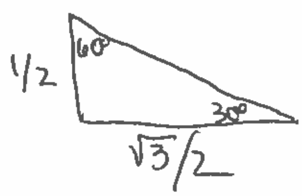
so

$\cos\left(\frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

$\sec\left(\frac{3\pi}{4}\right) = \frac{1}{-\sqrt{2}/2} = \frac{-2}{\sqrt{2}} = \boxed{-\sqrt{2}}$

(b) $\tan\left(\frac{5\pi}{6}\right) = \frac{\sin(5\pi/6)}{\cos(5\pi/6)}$, $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$





$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

$\tan\left(\frac{5\pi}{6}\right) = \frac{1/2}{-\sqrt{3}/2} = \boxed{\frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}}$

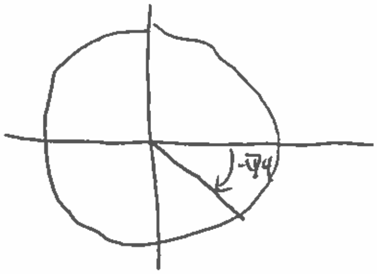
(c) $\csc\left(\frac{5\pi}{2}\right) = \frac{1}{\sin\left(\frac{5\pi}{2}\right)}$, $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$

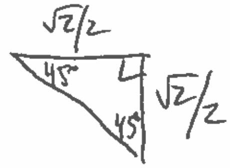
So $\sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 2\pi\right) = \sin\left(\frac{\pi}{2}\right)$

$= 1$

So $\csc\left(\frac{5\pi}{4}\right) = \frac{1}{1} = \boxed{1}$

B1. (d) $\cot(-\pi/4) = \dfrac{\cos(-\pi/4)}{\sin(-\pi/4)}$





$$\cos\left(\frac{-\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{-\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos(-\pi/4) = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = \boxed{-1}.$$

B2. (a) $\tan^2(x)\sec(x)\csc(x) = \left(\dfrac{\sin(x)}{\cos(x)}\right)^2 \cdot \dfrac{1}{\cos(x)} \cdot \dfrac{1}{\sin(x)}$

$$= \frac{\sin^{\cancel{2}}(x)}{\cos^3(x)\cancel{\sin(x)}} = \boxed{\frac{\sin(x)}{\cos^3(x)}}$$

(b) $\dfrac{\csc(x)\cot(x)}{\tan^2(x)\sec(x)} = \csc(x)\cdot\cot(x)\cdot\dfrac{1}{\tan^2(x)}\cdot\dfrac{1}{\sec(x)}$

$\sec(x) = \dfrac{1}{\cos(x)}$ so $\dfrac{1}{\sec(x)} = \cos$

$\tan(x) = \dfrac{1}{\cot(x)}$ so $\dfrac{1}{\tan(x)} = \cot(x)$

$$= \csc(x)\cdot\cot(x)\cdot\cot^2(x)\cdot\cos(x)$$

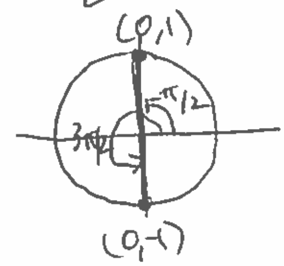
$$= \frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)}\cdot\frac{\cos^2(x)}{\sin^2(x)}\cdot\cos(x)$$

$$= \boxed{\frac{\cos^4(x)}{\sin^4(x)}}$$

B3. (a) $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\sec(x) = \dfrac{1}{\cos(x)}$, so

$\tan(x)$ and $\sec(x)$ are both undefined when $\cos(x) = 0$.

For $0 \le x < 2\pi$, $\cos(x) = 0$ when $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

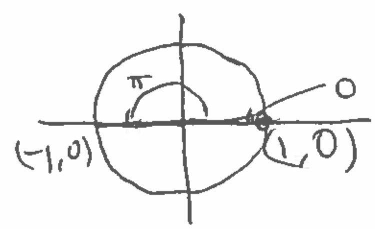


Since $\cos(x + 2\pi k) = \cos(x)$ for any integer k, then $\cos(x) = 0$ also for $x = \frac{\pi}{2} + 2\pi k$ or $\frac{3\pi}{2} + 2\pi k$ for any integer k.

So $\tan(x)$ and $\sec(x)$ are undefined when

$$\boxed{x = \frac{\pi}{2} + 2\pi k \text{ or } x = \frac{3\pi}{2} + 2\pi k \text{ for any integer } k}$$

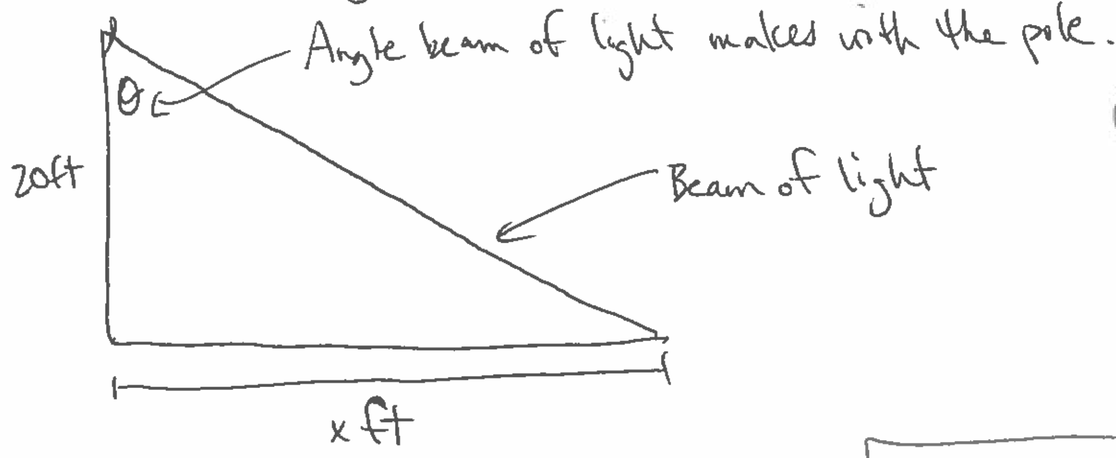
(b) $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ and $\csc(x) = \dfrac{1}{\sin(x)}$, so $\cot(x)$ and $\csc(x)$ are undefined when $\sin(x) = 0$. For $0 \le x < 2\pi$, $\sin(x) = 0$ for $x = 0$ or π.



Since also $\sin(x + 2\pi k) = \sin(x)$ for any integer k, we have $\cot(x)$ and $\csc(x)$ are undefined when:

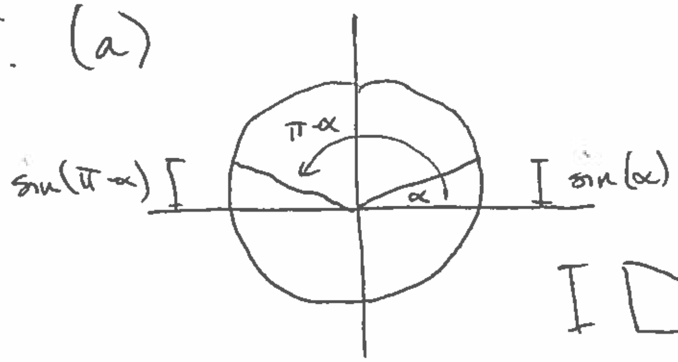
$$\boxed{x = 2\pi k \text{ or } \pi + 2\pi k \text{ for any integer } k.}$$

B4. The picture of light beam and pole are as follows:



From the picture, we have $\tan(\theta) = \frac{x}{20}$, so $\boxed{x = 20\tan(\theta)}$

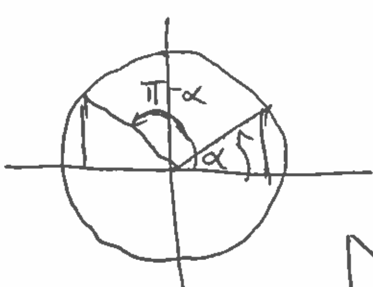
B5. (a)



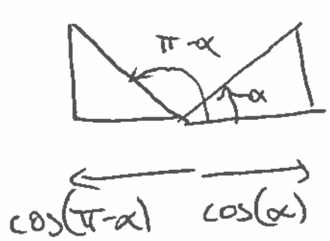
From the picture, the angles α and $\pi - \alpha$ correspond to right triangles

in the unit circle which have the same height, the first being $\sin(\alpha)$, and the second $\sin(\pi-\alpha)$. So $\sin(\alpha) = \sin(\pi - \alpha)$. For $\alpha = \frac{\pi}{6}$, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, and $\sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

(b)



Using a similar picture, the <u>bases</u> of the right triangles have the same length, but the coordinates are in opposite directions, so $\cos(\pi-\alpha) = -\cos(\alpha)$.



If $\alpha = \frac{\pi}{6}$, $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\pi - \frac{\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

B5 (c) Since $\tan(\pi-\alpha) = \dfrac{\sin(\pi-\alpha)}{\cos(\pi-\alpha)}$, and

from (a) and (b), $\sin(\pi-\alpha) = \sin(\alpha)$,

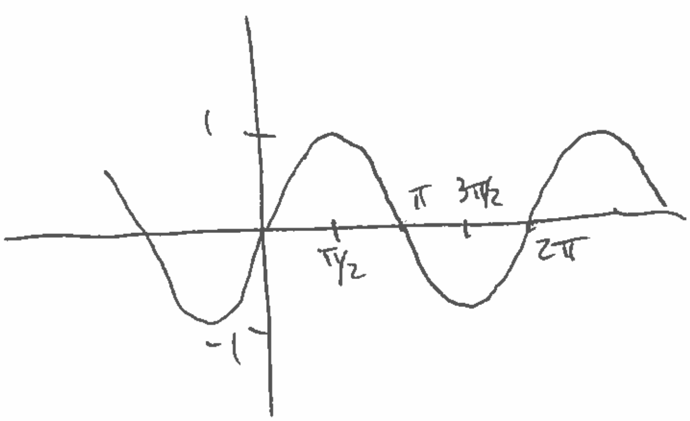
$\cos(\pi-\alpha) = -\cos\alpha$, then $\dfrac{\sin(\pi-\alpha)}{\cos(\pi-\alpha)} = \dfrac{\sin(\alpha)}{-\cos(\alpha)}$.

So $\tan(\pi-\alpha) = \dfrac{\sin(\alpha)}{-\cos(\alpha)} = -\dfrac{\sin(\alpha)}{\cos(\alpha)} = -\tan(\alpha)$.

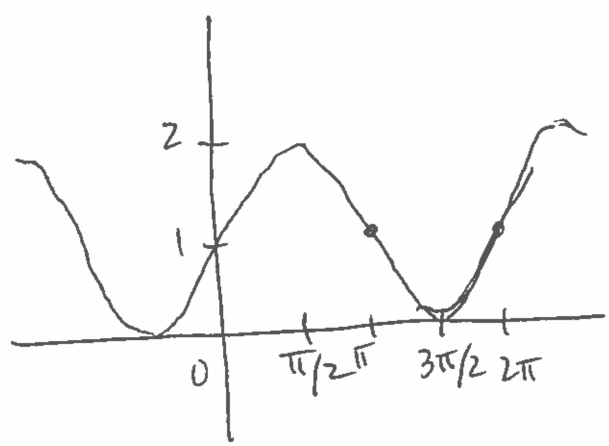
$$\boxed{\tan(\pi-\alpha) = -\tan(\alpha)}$$

B6. We have seen that the graph of $y = f(x) + a$ is the graph of $y = f(x)$ shifted up a units if $a > 0$, or down a units if $a < 0$.

(a) $y = \sin(x)$

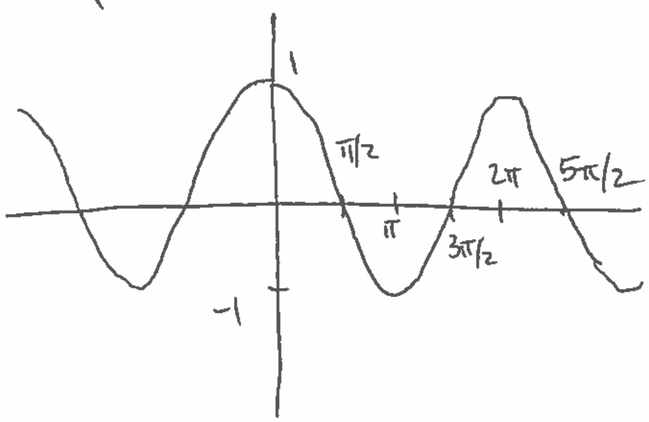


$y = \sin(x) + 1$

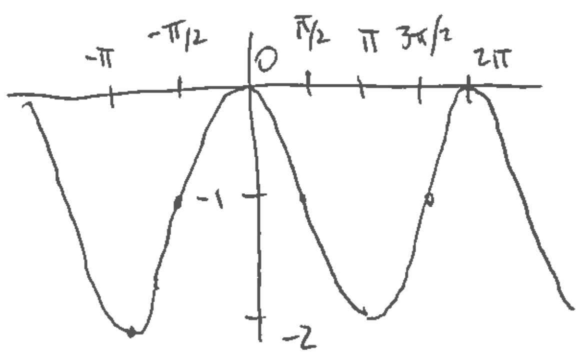


Also, the graph of $y = f(x-c)$ is the graph of $y = f(x)$ shifted to the right c units if $c > 0$ (or to the left if $c < 0$)

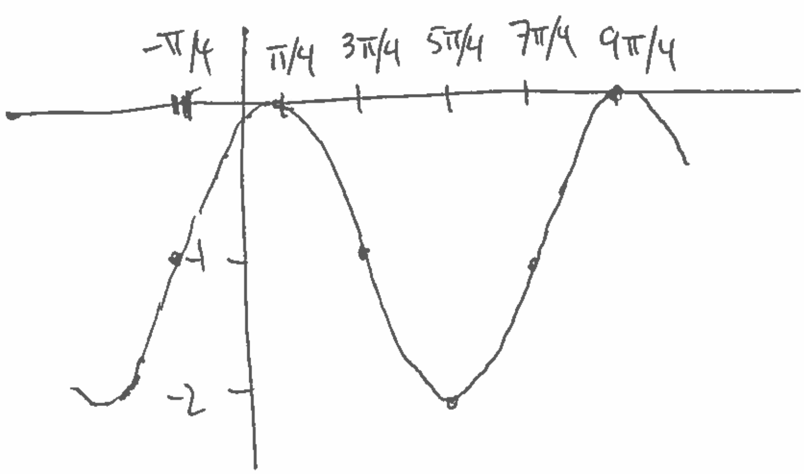
B6 (b) The graph of $y = \cos(x - \frac{\pi}{4}) - 1$ can be obtained by first graphing $y = \cos(x)$, then shifting it down one unit to obtain the graph of $y = \cos(x) - 1$, then shifting that to the right $\frac{\pi}{4}$ units to finally have the graph of $y = \cos(x - \frac{\pi}{4}) - 1$:



$y = \cos x$



$y = \cos(x) - 1$



$y = \cos\left(x - \frac{\pi}{4}\right) - 1$