Math 103 - HW #4 A, B

A1. $4^{1/4} \cdot \left(\frac{1}{2}\right)^{-3} = (2^2)^{1/4}(2^{-1})^{-3} = 2^{2/4} \cdot 2^3 =$

$= 2^{1/2} \cdot 2^3 = \boxed{2^{7/2}}$

A2. $\dfrac{9^{1/6} \cdot 3^{1/2}}{3^{-3} \cdot 3^4} = \dfrac{(3^2)^{1/6} \cdot 3^{1/2}}{3^{(-3+4)}} = \dfrac{3^{2/6} \cdot 3^{1/2}}{3^1} = 3^{1/3} \cdot 3^{1/2} \cdot 3^{-1} =$

$= 3^{\frac{2}{6}+\frac{3}{6}-1} = 3^{\frac{5}{6}-1} = \boxed{3^{-1/6}}$

A3. $(1000)^{1/3} \cdot 10^7 \cdot (10^2)^3 \cdot (100)^{-1/2} =$

$= (10^3)^{1/3} \cdot 10^7 \cdot 10^6 \cdot (10^2)^{-1/2} = 10^1 \cdot 10^7 \cdot 10^6 \cdot 10^{-1} =$

$= \boxed{10^{13}}$

A4. $\sqrt[5]{b^4} \cdot (b^2)^{3/5} \cdot b^{-3} = (b^4)^{1/5} \cdot (b^2)^{3/5} \cdot b^{-3} =$

$= b^{4/5} \cdot b^{6/5} \cdot b^{-3} = b^{10/5} \cdot b^{-3} = b^2 \cdot b^{-3} = \boxed{b^{-1}}$

If $b = 0$, then $b^{-1} = \frac{1}{b}$ is undefined

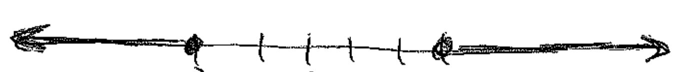
A5. $\left(\dfrac{\sqrt[3]{(x^2+1)^2}}{\sqrt{(x^2+1)^3}}\cdot(x^2+1)^{1/3}\right)^2 = \left(\dfrac{((x^2+1)^2)^{1/3}}{((x^2+1)^3)^{1/2}}\cdot(x^2+1)^{1/3}\right)^2 =$

$= \left(\dfrac{(x^2+1)^{2/3}}{(x^2+1)^{3/2}}\cdot(x^2+1)^{1/3}\right)^2 = \left((x^2+1)^{2/3}\cdot(x^2+1)^{-3/2}\cdot(x^2+1)^{1/3}\right)^2 =$

$= \left((x^2+1)^{2/3-3/2+1/3}\right)^2 = \left((x^2+1)^{-1/2}\right)^2 = \boxed{(x^2+1)^{-1}}$.

A6. We have $(x^2-x-6)^{1/4} = \sqrt[4]{x^2-x-6}$. The 4th root (or any <u>even</u> root) is only defined for <u>non-negative</u> numbers (otherwise the expression is not a real number). That is, $\sqrt[4]{x^2-x-6}$ is defined exactly when $x^2-x-6 \geq 0$, so when $(x-3)(x+2) \geq 0$. This is true when either $x-3 \geq 0$ and $x+2 \geq 0$, or, $x-3 \leq 0$ and $x+2 \leq 0$, so when either $x \geq 3$ and $x \geq -2$, or, $x \leq 3$ and $x \leq -2$. This means when $x \geq 3$ or $x \leq -2$. So the expression $(x^2-x-6)^{1/4}$ is defined when $x \geq 3$ or $x \leq -2$.

On the number line:



A7. Note that $(x^{1/2})^2 = x$, so we can write the equation $x - 2x^{1/2} - 8 = 0$ as $(x^{1/2})^2 - 2x^{1/2} - 8 = 0$. If we think of $z = x^{1/2}$, this becomes $z^2 - 2z - 8 = 0$, or $(z-4)(z+2) = 0$. That is, the original equation factors as $(x^{1/2} - 4)(x^{1/2} + 2) = 0$, or $(\sqrt{x} - 4)(\sqrt{x} + 2) = 0$. This expression is 0 when $\sqrt{x} - 4 = 0$ or $\sqrt{x} + 2 = 0$, so when $\sqrt{x} = 4$ or $\sqrt{x} = -2$.

$\sqrt{x} = 4$ when $(\sqrt{x})^2 = 4^2$, so $x = 16$.
However, $\sqrt{x} = -2$ can never occur, because $\sqrt{x}$ means the positive square root of x, so $\sqrt{x} \geq 0$ always (that is, $\sqrt{x}$ is never negative). So, the only solution is $\boxed{x = 16}$.

A8. (a): Similar to A6, the expression $\sqrt[6]{x-2}$ is only defined when $x - 2 \geq 0$, since we are taking an even root. The expression is only defined (as a real number) for $x \geq 2$, so for example if $x = 1$, then $\sqrt[6]{x-2} = \sqrt[6]{-1}$ is not a real number. So the statement is FALSE.

A8 (b): We compute the expression under the square root: $(x^{1/2} - x^{-1/2})^2 + 4 = (x^{1/2} - x^{-1/2})(x^{1/2} - x^{-1/2}) + 4 =$

$= (x^{1/2})^2 - (x^{-1/2})(x^{1/2}) - (x^{1/2})(x^{-1/2}) + (-x^{-1/2})^2 + 4$

$= x - x^{-1/2+1/2} - x^{1/2-1/2} + x^{-1} + 4 = x - x^0 - x^0 + x^{-1} + 4$

$= x - 2 + x^{-1} + 4 = x + 2 + x^{-1}.$

If we square the right side, we have:

$(x^{1/2} + x^{-1/2})^2 = (x^{1/2} + x^{-1/2})(x^{1/2} + x^{-1/2}) =$

$= (x^{1/2})^2 + (x^{-1/2})(x^{1/2}) + (x^{1/2})(x^{-1/2}) + (x^{-1/2})^2$

$= x^1 + x^0 + x^0 + x^{-1} = x + 2 + x^{-1}.$

But now we have $(x^{1/2} + x^{-1/2})^2 = (x^{1/2} - x^{-1/2})^2 + 4$, since they are both equal to $x + 2 + x^{-1/2}$.

Taking the (positive) square root of both sides gives $\sqrt{(x^{1/2} - x^{-1/2})^2 + 4} = x^{1/2} + x^{-1/2}$, so the statement is $\underline{\text{TRUE}}$.

B1. (a) $100^{3/2} = 1000$ means $\boxed{\log_{100} 1000 = \frac{3}{2}}$.

(b) $\left(\frac{1}{2}\right)^{-3} = 8$ means $\boxed{\log_{1/2} 8 = -3}$.

(c) $\left(\frac{9}{16}\right)^{-1/2} = \frac{4}{3}$ means $\boxed{\log_{9/16}\left(\frac{4}{3}\right) = -\frac{1}{2}}$.

(d) $\log_5 \sqrt[3]{25} = \frac{2}{3}$ means $\boxed{5^{2/3} = \sqrt[3]{25}}$.

(e) $\log_{100}(.001) = \frac{-3}{2}$ means $\boxed{100^{-3/2} = .001}$

(f) $\log_{2/3}\left(\frac{27}{8}\right) = -3$ means $\boxed{\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}}$.

B2. (a) $\log_2\left(\frac{1}{16}\right) = \boxed{-4}$ since $2^{-4} = \frac{1}{16} \left(= \frac{1}{2^4}\right)$.

(b) $\log_3 \sqrt[5]{27} = \log_3 \sqrt[5]{3^3} = \log_3 (3^3)^{1/5} = \log_3 3^{3/5} = \boxed{\frac{3}{5}}$.

(c) $\log_{10}(.00001) = \log_{10}(10^{-5}) = \boxed{-5}$.

(d) $\log_{3/4}\left(\sqrt[3]{\frac{16}{9}}\right) = \log_{3/4}\left(\sqrt[3]{\frac{4^2}{3^2}}\right) = \log_{3/4}\left(\left(\left(\frac{3}{4}\right)^{-2}\right)^{1/3}\right) = \boxed{\frac{-2}{3}}$

(e) $\log_2\left(\frac{1}{\sqrt{2}}\right) = \log_2\left(\frac{1}{2^{1/2}}\right) = \log_2\left(2^{-1/2}\right) = \boxed{\frac{-1}{2}}$.

(f) $2\log_x \sqrt[3]{x} - \frac{1}{3}\log(y^3) = 2\log_x(x^{1/3}) - \frac{1}{3}\cdot 3 = \frac{2}{3} - 1 = \boxed{\frac{-1}{3}}$

(g) $\log_5 50 - 2\log_5 2 + \log_5 10 = \log_5\left(\frac{50\cdot 10}{2^2}\right) = \log_5(\underbrace{25\cdot 5}_{=5^3}) = \boxed{3}$

B3. (a) $4\log_2(x^{1/2}) - \frac{3}{2}\log_2(x^4) + \log_2 x$

$$= \log_2\left((x^{1/2})^4\right) + \log_2\left((x^4)^{-3/2}\right) + \log_2 x$$

$$= \log_2(x^2) + \log_2(x^{-6}) + \log_2 x$$

$$= \log_2(x^2 \cdot x^{-6} \cdot x) = \boxed{\log_2(x^{-3})}.$$

(b) $\log_3\left(\frac{x^2}{y^3}\right) - 3\log_3(y^{-1}) - 2\log_3\left(\frac{y^{1/2}}{x^{-2}}\right)$

$$= \log_3\left(\frac{x^2}{y^3}\right) + \log_3\left((y^{-1})^{-3}\right) + \log_3\left(\left(\frac{y^{1/2}}{x^{-2}}\right)^{-2}\right)$$

$$= \log_3\left(\frac{x^2}{y^3}\right) + \log_3(y^3) + \log_3\left(\frac{y^{-1}}{x^4}\right)$$

$$= \log_3\left(\frac{x^2}{y^3} \cdot y^3 \cdot \frac{y^{-1}}{x^4}\right) = \boxed{\log_3\left(\frac{1}{x^2 y}\right)}$$

B4. $\log_3(2x-4) = 2$ means $3^2 = 2x-4$, so

$2x-4 = 9$, so $2x = 13$, so $\boxed{x = \frac{13}{2}}$.

B5. $16 \cdot 2^x = 4^{13}$ can be written as:

$2^4 \cdot 2^x = (2^2)^{13}$, or $2^{4+x} = 2^{26}$. Taking $\log_2$ of both sides gives $\log_2(2^{4+x}) = \log_2(2^{26})$, so

$4 + x = 26$, giving $\boxed{x = 22}$.

B6. $\log_2(x^2-2x+3)=2$ means $2^2=x^2-2x+3$,

so $x^2-2x+3=4$, or $x^2-2x-1=0$.

We can't factor, so we use the quadratic formula with $a=1, b=-2, c=-1$ (so $b^2-4ac=4-4(1)(-1)=8\geq 0$).

We get $x=\dfrac{2\pm\sqrt{4-4(1)(-1)}}{2}=\dfrac{2\pm\sqrt{8}}{2}=\dfrac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$.

The solutions are $\boxed{x=1+\sqrt{2} \text{ or } 1-\sqrt{2}}$.

B7. $b^{-7x}=5My^3$. To solve for x, first take $\log_b$ of both sides: $\log_b(b^{-7x})=\log_b(5My^3)$.

Since $\log_b(b^{-7x})=-7x$, we have $-7x=\log_b(5My^3)$.

Thus $\boxed{x=\frac{-1}{7}\log_b(5My^3)}$.

B8. If $27^y=9$, then $(3^3)^y=3^2$, so $3^{3y}=3^2$. Taking $\log_3$ of both sides gives $3y=2$, so $y=\frac{2}{3}$.

Now $8^{3x-2y}=4$ means $2^{3(3x-2y)}=2^2$, so $2^{9x-6y}=2^2$. Taking $\log_2$ of both sides gives $9x-6y=2$. Since $y=\frac{2}{3}$, $9x-6\left(\frac{2}{3}\right)=2$, so $9x-4=2$, so $9x=6$, so $\boxed{x=\frac{2}{3}}$.