Math 103 - HW#3

A1. (a): $(x^2+2)(x^2+x+1) = x^2\cdot x^2 + x^2\cdot x + x^2\cdot 1 + 2x^2+2x+2$

$$= \boxed{x^4+x^3+3x^2+2x+2}$$

(b): $(x^2+2x-1)(x^3+3) = x^2\cdot x^3 + x^2\cdot 3 + 2x\cdot x^3 + 2x\cdot 3 - x^3 - 3$

$$= \boxed{x^5+2x^4-x^3+3x^2+6x-3}$$

(c): $(a-b)(a^2+ab+b^2) = a\cdot a^2 + a\cdot ab + a\cdot b^2 - b\cdot a^2 - b\cdot ab - b\cdot b^2$

$$= a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 = \boxed{a^3-b^3}$$

(d): $(a+b)(a^2-ab+b^2) = a\cdot a^2 - a\cdot ab + a\cdot b^2 + b\cdot a^2 - b\cdot ab$

$$+ b\cdot b^2 = a^3 - \cancel{a^2b} + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + b^3 = \boxed{a^3+b^3}$$

A2. (a) If we use long division here, it looks like:

$$
\begin{array}{r l}
 & 2x^8 - 5x^4 + 2x - 3 \\
x+2 \,\big) & 2x^9+4x^8+0x^7+0x^6-5x^5-10x^4+0x^3+2x^2+x-6 \\
 & -(2x^9+4x^8) \\
\hline
 & -5x^5-10x^4+0x^3+2x^2+x-6 \\
 & -(-5x^5-10x^4) \\
\hline
 & 2x^2+x-6 \\
 & -(2x^2+4x) \\
\hline
 & -3x-6 \\
 & -(-3x-6) \\
\hline
 & 0
\end{array}
$$

A2. (a) (cont'd) If we use synthetic division, it looks like:

$$
\begin{array}{r|rrrrrrrrrr}
-2 & 2 & 4 & 0 & 0 & -5 & -10 & 0 & 2 & 1 & -6 \\
 & & -4 & 0 & 0 & 0 & 10 & 0 & 0 & -4 & 6 \\
\hline
 & 2 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & -3 & \boxed{0}
\end{array}
$$

In either case, we find the quotient is:

$$\boxed{2x^8 - 5x^4 + 2x - 3}$$

(b)

$$
\begin{array}{r}
x^3 - 4 \\
x^3 + 2x + 1 \,\big)\, \overline{x^6 + 0x^5 + 2x^4 - 3x^3 + 0x^2 - 8x - 4} \\
-(x^6 + 2x^4 + x^3) \\
\hline
-4x^3 + 0x^2 - 8x - 4 \\
-(-4x^3 - 8x - 4) \\
\hline
0
\end{array}
$$

So the quotient is $\boxed{x^3 - 4}$

(c)

$$
\begin{array}{r}
x^5 - 3x^3 + 2x - 1 \\
x^2 + 3 \,\big)\, \overline{x^7 + 0x^6 + 0x^5 + 0x^4 - 7x^3 - x^2 + 6x - 3} \\
-(x^7 + 3x^5) \\
\hline
-3x^5 + 0x^4 - 7x^3 - x^2 + 6x - 3 \\
-(-3x^5 - 9x^3) \\
\hline
2x^3 - x^2 + 6x - 3 \\
-(2x^3 + 6x) \\
\hline
-x^2 - 3 \\
-x^2 - 3 \\
\hline
0
\end{array}
$$

So the quotient is

$$\boxed{x^5 - 3x^3 + 2x - 1}$$

A3. To check $x=5$ is a solution to $x^3-3x^2-13x+15=0$ using synthetic division looks like:

$$\begin{array}{r|rrrr} 5 & 1 & -3 & -13 & 15 \\ & & 5 & 10 & -15 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

The result gives us a quotient of x^2+2x-3.

$\leftarrow$ (with 0 remainder)

This means $x-5$ is a factor, and we have:

$$x^3-3x^2-13x+15=(x-5)(x^2+2x-3)=0.$$

Since $x^2+2x-3=(x+3)(x-1)$, we have:

$$x^3-3x^2-13x+15=(x-5)(x+3)(x-1)=0$$

So $x-5=0$, $x+3=0$, or $x-1=0$. The solutions are thus $\boxed{x=5,-3,\text{ or } 1}$.

A4. To ~~a~~ search for one solution to $x^3-3x+2=0$, we try factors of the constant term 2 through synthetic division. So, we try $1, -1, 2, -2$.

It turns out $x=1$ works:

(Don't forget the $0x^2$!)

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

So $x-1$ is a factor, and the quotient is

$$x^2+x-2$$

A4 (cont'd) So we have

$x^3-3x+2=(x-1)(x^2+x-2)=0$. Since

$x^2+x-2=(x+2)(x-1)$, we have

$x^3-3x+2=(x-1)(x+2)(x-1)=(x-1)^2(x+2)=0$.

Thus $(x-1)^2=0$, so $x-1=0$, or $x+2=0$.

The only solutions are $\boxed{x=1 \text{ or } -2}$

A5. We first compute the long division:

$$
\begin{array}{r|l}
 & x^2+2x+7 \\
\hline
x^2+x+1 & x^4+3x^3+10x^2+9x+7 \\
 & -(x^4+x^3+x^2) \\
\hline
 & 2x^3+9x^2+9x+7 \\
 & -(2x^3+2x^2+2x\ 1 \\
\hline
 & 7x^2+7x+7 \\
 & -(7x^2+7x+7) \\
\hline
 & 0
\end{array}
$$

So we have $x^4+3x^3+10x^2+9x+7$

$$=(x^2+x+1)(x^2+2x+7)=0.$$

This means either $x^2+x+1=0$ or $x^2+2x+7=0$

If we try to factor these, we find we cannot.

So we turn to the quadratic formula.

A5. (cont'd)

In the first, $x^2+x+1=0$, $a=1, b=1, c=1$. So $b^2-4ac = 1^2-4(1)(1) = -3 < 0$. Since the discriminant is negative, $\sqrt{b^2-4ac}$ is not a real number and there are no solutions. In the second, $x^2+2x+7=0$, $a=1, b=2, c=7$, so $b^2-4ac = 2^2-4(1)(7) = -24 < 0$. Again, there are no real solutions.

Since $x^4+3x^3+10x^2+9x+7 = (x^2+x+1)(x^2+2x+7)$, but $x^2+x+1=0$ and $x^2+2x+7=0$ have no real solutions, then $x^4+3x^3+10x^2+9x+7=0$ has no real solutions.

A6. The expression $\dfrac{x^8-x^2+2x-5}{x^3-x^2-9x+9}$ is undefined exactly for those x when this is division by 0. So, it is undefined exactly when $x^3-x^2-9x+9=0$. To find solutions, we begin like we did in A4, trying to find one solution through synthetic division, and trying factors of the constant term 9.

So, we try $1, -1, 3, -3, 9, -9$.

After maybe trying a few of these, we find $x=3$ works:
(or maybe you find $x=1$ works first!)

A6: (cont'd)

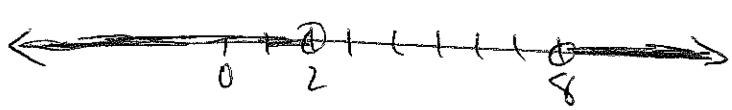
$$\begin{array}{r|rrrr} 3 & 1 & -1 & -9 & 9 \\ & & 3 & 6 & -9 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

So, $x-3$ is a factor, with quotient x^2+2x-3.

Since $x^2+2x-3=(x+3)(x-1)$, we have $x^3-x^2-9x+9=(x-3)(x+3)(x-1)$, so the expression is undefined when $x^3-x^2-9x+9=(x-3)(x+3)(x-1)=0$, so when $\boxed{x=3,\ -3,\ \text{or}\ 1}$.

A7: $x^2-10x+16=(x-8)(x-2)$, so $x^2-10x+16=0$ when $x-8=0$ or $x-2=0$, so when $\boxed{x=8 \text{ or } x=2}$.

$x^2-10x+16=(x-8)(x-2)>0$ when either $x-8>0$ and $x-2>0$, or, $x-8<0$ and $x-2<0$, so $x>8$ and $x>2$, or, $x<8$ and $x<2$, so when $\boxed{x>8 \text{ or } x<2}$. On the number line, this looks like:

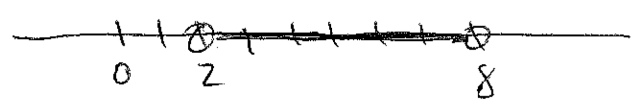


A7 (cont'd) $x^2-10x+16<0$ when $(x-8)(x-2)<0$, so when either

$x-8<0$ and $x-2>0$, or, $x-8>0$ and $x-2<0$,

so $x<8$ and $x>2$, or, $x>8$ and $x<2$, which is impossible.

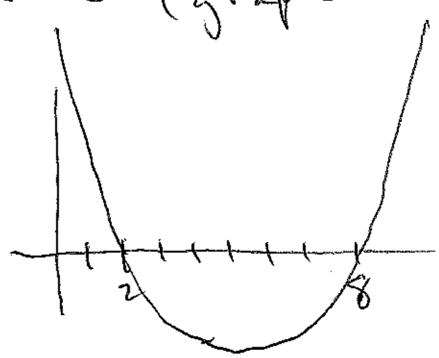
So $(x-8)(x-2)<0$ when $x<8$ and $x>2$, which can be written as $\boxed{2<x<8}$. On the number line this looks like:



(Note: Since we found, before this, all x such that $x^2-10x+16\geq 0$, we could also argue that for all <u>other</u> x we must have $x^2-10x+16<0$, which still gives us all x such that $2<x<8$).

For $y=x^2-10x+16$, we know $y=0$ when $x=2$ or 8, $y>0$ when $x<2$ or $x>8$ (graph is above the x-axis), and $y<0$ when $2<x<8$ (graph is below the x-axis).

The graph will thus look something like:



B1:

$$
\begin{array}{r|l}
 & 3 \leftarrow \text{(quotient)} \\
\hline
x^2-3x+2 & 3x^2+2x+1 \\
 & -(3x^2-9x+6) \\
\hline
 & 11x-5 \leftarrow \text{(remainder)}
\end{array}
$$

So $\boxed{\dfrac{3x^2+2x+1}{x^2-3x+2} = 3 + \dfrac{11x-5}{x^2-3x+2}}$

B2:

$$
\begin{array}{r|l}
 & 2x^3-4x^2+7x-12 \\
\hline
x+2 & 2x^4+0x^3-x^2+2x-3 \\
 & -(2x^4+4x^3) \\
\hline
 & -4x^3-x^2+2x-3 \\
 & -(-4x^3-8x^2) \\
\hline
 & 7x^2+2x-3 \\
 & -(7x^2+14x) \\
\hline
 & -12x-3 \\
 & -(-12x-24) \\
\hline
 & 21
\end{array}
$$

So $\dfrac{2x^4-x^2+2x-3}{x+2} = \boxed{2x^3-4x^2+7x-12 + \dfrac{21}{x+2}}$

B3:

$$
\begin{array}{r|l}
 & x^2 \\
\hline
x^3-1 & x^5+0x^4+0x^3+0x^2+0x+1 \\
 & -(x^5 - x^2) \\
\hline
 & x^2+1
\end{array}
$$

So: $\boxed{\dfrac{x^5+1}{x^3-1} = x^2 + \dfrac{x^2+1}{x^3-1}}$

B4:

$$\begin{array}{r} x^2 - 1 \\ x^3+x+1 \overline{\smash{)}\, x^5 + 0x^4 + 0x^3 + 0x^2 + 2x + 1} \\ -(x^5 \quad\quad + x^3 + x^2) \\ \hline -x^3 - x^2 + 2x + 1 \\ -(-x^3 \quad\quad - x - 1) \\ \hline -x^2 + 3x + 2 \end{array}$$

So $\boxed{\dfrac{x^5+2x+1}{x^3+x+1} = x^2 - 1 + \dfrac{-x^2+3x+2}{x^3+x+1}}$.

B5 (a): We first find all x such that

$x^2 - 12x + 27 = 0$, so $(x-9)(x-3) = 0$.

So $x = 9$ or $x = 3$. We have to check that these do not make the denominator 0, otherwise the expression is undefined.

The denominator is $x^3 + 5x + 2$. Note for x positive, this expression is always positive, so cannot be 0 for $x = 9$ or $x = 3$. Just to make sure, we can plug these in:

$9^3 + 5(9) + 2 = 243 + 45 + 2 = 290 \neq 0$,

$3^3 + 5(3) + 2 = 27 + 15 + 2 = 44 \neq 0$.

So the values making the rational expression 0 are $\boxed{x = 9, 3}$.

B5 (b): We first find all x making $x^3+2x^2+x-4=0$. We use synthetic division to find 1 solution, then we go from there. Trying $x=1$ gives:

$$\begin{array}{r|rrrr} 1 & 1 & 2 & 1 & -4 \\ & & 1 & 3 & 4 \\ \hline & 1 & 3 & 4 & \boxed{0} \end{array}$$

This works, and gives $x^3+2x^2+x-4=(x-1)(x^2+3x+4)$.

Now, $x^2+3x+4=0$ cannot be solved by factoring, so we use the quadratic formula with $a=1$, $b=3$, $c=4$. Checking b^2-4ac gives $3^2-4(1)(4)=-5<0$, so this has no solutions. So, $x^3+2x^2+x-4=0$ when $(x-1)(x^2+3x+4)=0$, so when $x=1$ or when $x^2+3x+4=0$, which has no solutions. So $x=1$ is the only solution.

Finally, we check that $x=1$ does not make the denominator 0: $1^4+1+1=3\neq 0$. So the only value making this rational expression 0 is $\boxed{x=1}$.

B5 (c): The numerator is 0 when $x^2-x-2=0$, so when $(x-2)(x+1)=0$, so when $x=2$ or $x=-1$. We check if these make the denominator 0:

$2^3+2^2+2+1=15\neq 0$, for $x=2$, while for $x=-1$, $(-1)^3+(-1)^2+(-1)+1=-1+1-1+1=0$.

So $x=-1$ makes the expression undefined since the denominator is 0. The only value making the rational expression 0 is $\boxed{x=2}$.

B6: If $2+\frac{4}{x^2+2x-1}=0$, then $\frac{4}{x^2+2x-1}=-2$, so (dividing by 2) $\frac{2}{x^2+2x-1}=-1$, and multiplying by x^2+2x-1 on both sides (assuming it is not 0, which we check in the next part) gives $2=-x^2-2x+1$.

Bringing everything ~~to~~ to the left side gives us $x^2+2x+1=0$, or $(x+1)(x+1)=0$. The only value making this 0 is $\boxed{x=-1}$. Note also that when $x=-1$, $x^2+2x-1=(-1)^2-2-1=-2\neq 0$, so the expression is defined for this value.

B6 (cont'd) The expression $2+\frac{4}{x^2+2x-1}$ is undefined when the denominator is 0, so when $x^2+2x-1=0$. We can't factor, so we use the quadratic formula with $a=1$, $b=2$, $c=-1$. We have $b^2-4ac = 2^2-4(1)(-1) =$ $= 8 \geq 0$, so there are solutions. The solutions are:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}.$$

So the values which make the expression undefined are $\boxed{x = -1+\sqrt{2}, -1-\sqrt{2}}$.

B7 $\frac{x-2}{x-1} \geq 0$ when either both the top and bottom are positive, or both are negative (very similar to when $(x-2)(x-1) > 0$). So, this happens when either $x-2>0$ and $x-1>0$, or, $x-2<0$ and $x-1<0$, so when $x>2$ and $x>1$, or, $x<2$ and $x<1$, so

$$\boxed{x > 2 \quad \text{or} \quad x < 1}.$$

On the number line, this looks like: