Polynomial discussion problems

1. Find a polynomial $P(x)$ such that $P(x)$ is divisible by x^2+1 and $P(x)+1$ is divisible by x^3+x^2+1.

2. Let $P(x)$ be a polynomial leaving the remainder A when divided by $x-a$ and the remainder B when divided by $x-b$, $a \neq b$. Find the remainder when $f(x)$ is divided by $(x-a)(x-b)$.

3. Find polynomials $F(x)$ and $G(x)$ such that $(x^8-1)F(x)+(x^5-1)G(x)=x-1$. (Note: It is always possible to find such F, G since $\gcd(x^8-1, x^5-1)$=$x-1$.)

4. (Putnam 1963-B1) For which integer a does x^2-x+a divide $x^{13}+x+90$? (Hint: 90 must be divisible by a.)

5. Let $P(x)$ be a polynomial with integer coefficients.

   (a) Show that for integers a and b, if $a \equiv b \pmod{k}$, then $F(a) \equiv F(b) \pmod{k}$.

   (b) (Putnam 1940-A1) Show that if there exists an integer k such that none of the integers $F(1)$, $F(2), \cdots, F(k)$ is divisible by k, then $F(x)$ has no integer zero.