Problem Set 6

Discussion: Nov. 6

Discussion Problems

1. (a) Prove that for $a, b, c > 0$ satisfying $(1+a)(1+b)(1+c) = 8$, then $abc \leq 1$.
   (b) Prove that for $a, b, c > 0$, then $(a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \geq 9a^2b^2c^2$. (`Carolyn`)

2. Given that a, b, c, d, e are real numbers such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$. Find the maximum value of e. (`Alex`)

3. Prove that if a and b are positive numbers such that $a + b = 1$, then $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}$. (`Kassie`)

4. Let a, b, c denote the lengths of the sides of a triangle. Show that $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2$. (`Katelyn`)

5. (Putnam 1998-B1) Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

   for $x > 0$. (`Katie`)

6. Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999999}{1000000} < \frac{1}{1000}$$

   (Hint: square each side and "give a little" to create a "telescoping" product)(`Sean`)

7. (Putnam 2004-B2) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

   (`David`)

8. In sample problems, we proved: Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Show that $n(n+1)^{1/n} < n + H_n$ for every positive integer n. Now prove $H_n + (n-1)n^{-1/(n-1)} < n$ still using AM-GM inequality. (`Drew`)

More Problems:

1. (Putnam 1962-B5) Show that for $n > 1$, $\frac{3n+1}{2n+2} < \sum_{r=1}^{n} \frac{r^n}{n^n} < 2$.

2. (This is from the same book which has the selling chicken problem) Three pasture fields have areas of 7/3, 10 and 24 acres, respectively. The fields initially are covered with grass of the same thickness and new grass grows on each at the same rate per acre. If 12 cows eat the first field bare in 4 weeks and 212 cows eat the second field bare in 9 weeks, how many cows will eat the third field bare in 18 weeks? Assume that all cows eat at the same rate.

3. Find the remainder $1 \leq r < 13$, when 2^{1985} is divided by 13.

4. (Putnam 2004-A6) Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy \right)^2 dx$$
$$\leq \left(\int_0^1 \int_0^1 f(x,y)dx\, dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\, dx\, dy.$$

5. (Putnam 1996-B3) Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

6. (Putnam 1957-B3) Suppose that $f : [0, 1] \to R+$ is a monotonic decreasing function. Show that $\int_0^1 f(x)dx \cdot \int_0^1 x[f(x)]^2 dx \leq \int_0^1 xf(x)dx \cdot \int_0^1 [f(x)]^2 dx.$

7. (Putnam 2003-B6) Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \geq \int_0^1 |f(x)|\, dx.$$