Problem Set 5

Discussion Problems

1. (VT 1983) Let $f(x) = 1/x$ and $g(x) = 1 - x$ for $x \in (0,1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $(ax+b)/(cx+d)$, and prove that your list is exhaustive. (`Drew`) (Hint: use matrix)

2. (UIUC 2003 Mock) Let $f(x) = \dfrac{1}{1-x}$. Let $f_1(x) = f(x)$ and for each $n = 2, 3, \cdots$, let $f_n(x) = f(f_{n-1}(x))$. What is the value of $f_{2003}(2003)$? (`Carolyn`)

3. (UIUC 1997) Let $x_1 = x_2 = 1$, and $x_{n+1} = 1996x_n + 1997x_{n-1}$ for $n \geq 2$. Find (with proof) the reminder of x_{1997} upon division by 3. (`Alexander`) (Hint: (a) find the pattern; (b) find the periodic pattern modulo 3)

4. (UIUC 1997) Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \dfrac{x_n + nx_{n-1}}{n+1}$ for $n \geq 1$. Show that the sequence $\{x_n\}$ converges and finds its limit. (`Kassie`) (Hint: guess a general formula)

5. (VT 2005) We wish to tile a strip of n 1-inch by 1-inch squares. We can use dominos which are made up of two tiles which cover two adjacent squares, or 1-inch square tiles which cover one square. We may cover each square with one or two tiles and a tile can be above or below a domino on a square, but no part of a domino can be placed on any part of a different domino. We do not distinguish whether a domino is above or below a tile on a given square. Let $t(n)$ denote the number of ways the strip can be tiled according to the above rules. Thus for example, $t(1) = 2$ and $t(2) = 8$. Find a recurrence relation for $t(n)$, and use it to compute $t(6)$. (`Katelyn`)

6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 \neq 0$ and $a_{n+3} = 2a_{n+2} + 5a_{n+1} - 6a_n$. Find all possible values for $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$. (`Katie`) (Hint: find general solution, this is linear)

7. (Putnam 1958-A2) Define $a_1 = 1$, $a_{n+1} = 1 + \dfrac{n}{a_n}$. Show that $\sqrt{n} \leq a_n < 1 + \sqrt{n}$. (`Sean`)

8. $a_0 = 1$, $a_1 = 5$, $a_n = \dfrac{2a_{n-1}^2 - 3a_{n-1} - 9}{2a_{n-2}}$. Prove that a_n is an integer for all n. (`David`) (Hint: prove it is a 2nd order linear)

More Problems:

1. (Putnam 1980-B3) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (express the answer in the simplest form) (Hint: solve the recurrence)

2. (Putnam 1956-B6) $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. (a) Prove that any two terms in $\{a_n\}$ are relatively prime; (b) Prove that $\sum_{n=1}^{\infty} 1/a_n = 1$. (Hint: $b_n = a_n - 1$)

3. (Putnam 1993-A2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$. (Hint: prove $(x_{n+1} + x_{n-1})/x_n$ is a constant)

4. (Putnam 1970-A4) Given a sequence $\{x_n\}$, $n = 1, 2, \cdots$, such that $\lim\limits_{n\to\infty}(x_n - x_{n-2}) = 0$. Prove that $\lim\limits_{n\to\infty} \dfrac{x_n - x_{n-1}}{n} = 0$.

5. (Putnam 1966-A3) Let $0 < x_0 < 1$, and $x_{n+1} = x_n(1 - x_n)$ for $n \geq 0$. Prove that the limit $\lim\limits_{n\to\infty} nx_n$ exists and is equal to 1.

6. (Putnam 1969-B3) The terms of a sequence T_n satisfy $T_n T_{n+1} = n$ $(n = 1, 2, 3, \cdots)$ and $\lim_{n\to\infty} \frac{T_n}{T_{n+1}} = 1$. Show that $\pi T_1^2 = 2$.

7. (UIUC 1999) Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$, and $x_{n+1} = \sqrt{2}^{x_n}$ for $n \geq 1$. Prove the sequence $\{x_n\}$ converges and find its limit.

8. (Putnam 1994-A1) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

9. (Putnam 1997-A6) For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,
$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$
Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

10. (Putnam 1999-A6) The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,
$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$
Show that, for all n, a_n is an integer multiple of n.

11. (UIUC 1995) Let c be a positive constant, let $0 < x_1 < x_0 < 1$, and for $n \geq 1$ let $x_{n+1} = cx_n x_{n-1}$. Prove that there exists a positive real number α such that the limit $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n^\alpha}$ exists and $0 < L < \infty$.

12. (Putnam 1985-A3) Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition
$$a_m(0) = d/2^m,$$
$$a_m(j+1) = (a_m(j))^2 + 2a_m(j), \qquad j \geq 0.$$
Evaluate $\lim_{n\to\infty} a_n(n)$.