Problem Set 2

Discussion Problems Discussion: Sept 18

1. Prove that $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is an integer for $n = 0, 1, \cdots ,$. **(Katie)**
2. Show that, if n is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by n^2. **(Sean)**
3. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer. **(David)**
4. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5 cents and 12 cents stamps. Prove that we can perform the task if $n \geq 44$. **(Drew)**
5. Show that if a round robin tournament has an odd number of teams, it is possible for every team to win exactly half of its games. (hint: assuming the result for $2n - 1$ teams, show how the same thing could happen when 2 teams are added.) **(Carolyn)**
6. Prove that $1 < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} < 2$ for all positive integer n. **(Alexander)**
7. If each person, in a group of n people, is a friend of at least half the people in the group, then it is possible to seat the n people in a circle so that everyone sits next to friends only. **(Kassie)**
8. (VT, 2005-2) Find, and write out explicitly, a permutation $(p(1), p(2), \cdots, p(20))$ of (1,2, . . . ,20) such that k + p(k) is a power of 2 for k = 1,2, . . . , 20, and prove that only one such permutation exists. (To illustrate, a permutation of (1,2,3,4,5) such that k+ p(k) is a power of 2 for k = 1,2, . . . ,5 is clearly (1,2,5,4,3), because 1+1 = 2, 2+2 = 4, 3+5 = 8, 4+4 = 8, and 5+3 = 8.)**(Katelyn)**
9. (a) (VT, 2005-4) A 9×9 chess board has two squares from opposite corners and its central square removed (so 3 squares on the same diagonal are removed, leaving 78 squares). Is it possible to cover the remaining squares using dominoes, where each domino covers two adjacent squares? Justify your answer. (b) Given a $(2m + 1) \times (2n + 1)$ checkerboard where the four corner squares are black, show that if one removes any one red and two black squares, the remaining board can be covered with dominoes (1×2 rectangles).
10. (Putnam 2004 A3) Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Challenging Problems:

1. Show that among any 6 points in a 3×4 rectangle there is a pair of points not more than $\sqrt{5}$ apart.
2. (IMO 2005) In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than 2/5 of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.