Pigeonhole Principle

The pigeonhole principle states that if n pigeons are put into m pigeonholes, and if $n > m$, then at least one pigeonhole must contain more than one pigeon. Another way of stating this would be that m holes can hold at most m objects with one object to a hole; adding another object will force you to reuse one of the holes.

Although the pigeonhole principle may seem to be a trivial observation, it can be used to demonstrate unexpected results. For example, there must be at least two people in London with the same number of hairs on their heads. Demonstration: a typical head of hair has around $150,000$ hairs. It is reasonable to assume that no-one has more than $1,000,000$ hairs on their head. There are more than $1,000,000$ people in London. If we assign a pigeonhole for each number of hairs on a head, and assign people to the pigeonhole with their number of hairs on it, there must be two people with the same number of hairs on their heads.

A generalized version of this principle states that, if n discrete objects are to be allocated to m containers, then at least one container must hold no fewer than $\lceil n/m \rceil$ objects, where $\lceil \cdots \rceil$ denotes the ceiling function.

Dirichlet's Theorem: Let α be an irrational number. Then there are infinitely many integer pairs (h, k) where $k > 0$ such that

$$\left| \alpha - \frac{h}{k} \right| < \frac{1}{k^2}.$$

Examples:

1. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?
2. Given any $n+2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$?
3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 100$. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.]
4. Given any $n+1$ distinct integers between 1 and $2n$, show that two of them are relatively prime. Is this result best possible, *i.e.*, is the conclusion still true for n integers between 1 and $2n$?
5. Given any $n+1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, *i.e.*, is the conclusion still true for n integers between 1 and $2n$?
6. (Putnam 1980, A4(a)) Prove that there exist integers a, b, c not all zero and each of them absolute value less than one million, such that
$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$
7. Prove that there is some integer power of 2 that begins $2006\cdots$.
8. (Putnam 1980, A4(b)) (not related to pigeon hole principle) Let a, b, c be integers not all zero and each of them absolute value less than one million. Prove that
$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$
9. Given any $2n$ integers, show that there are n of them whose sum is divisible by n. (Though superficially similar to some other pigeonhole problems, this problem is much more difficult and does not really involve the pigeonhole principle.)