

## Principle of Mathematical induction (from Wikipedia)

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true of all natural numbers. The method can be extended to prove statements about more general well-founded structures, such as trees; this generalization, known as structural induction, is used in mathematical logic and computer science. Indeed, the validity of mathematical induction is logically equivalent to the well-ordering principle.

Mathematical induction is used to prove that every statement in an infinite sequence of statements is true. It is done by proving that the first statement in the infinite sequence of statements is true, and then proving that if any statement in the infinite sequence of statements is true, then so is the next one.

The simplest and most common form of mathematical induction proves that a statement holds for all natural numbers  $n$  and consists of two steps:

1. The basis: showing that the statement holds when  $n = 0$ .
2. The inductive step: showing that if the statement holds for  $n = m$ , then the same statement also holds for  $n = m + 1$ .

The proposition following the word “if” in the inductive step is called the induction hypothesis (or inductive hypothesis). To perform the inductive step, one assumes the induction hypothesis (that the statement is true for  $n = m$ ) and then uses this assumption to prove the statement for  $n = m + 1$ .

### Examples:

1. Prove that  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  for all positive integers  $n$ .
2. Prove that  $\frac{n^n}{3^n} < n! < \frac{n^n}{2^n}$  for  $n \geq 6$ .
3. George Pólya posed the following exercise: Find the error in the following argument, which purports to prove by mathematical induction that all horses are of the same color:  
*If there's only one horse, there's only one color. Suppose within any set of  $n$  horses, there is only one color. Now look at any set of  $n + 1$  horses. Number them:  $1, 2, 3, \dots, n, n + 1$ . Consider the sets  $\{1, 2, 3, \dots, n\}$  and  $\{2, 3, 4, \dots, n + 1\}$ . Each is a set of only  $n$  horses, therefore with each there is only one color. But the two sets overlap, so there must be only one color among all  $n + 1$  horses.*
4. A group of  $n$  people play a round-robin tournament. Each game ends in either a win or a loss. Show that it is possible to label the players  $P_1, P_2, P_3, \dots, P_n$  in such a way that  $P_1$  defeated  $P_2$ ,  $P_2$  defeated  $P_3$ , ... ,  $P_{n-1}$  defeated  $P_n$ .
5. If one square of a  $2^n \times 2^n$  chessboard is removed, then the remaining board can be covered by  $L$ -shaped trominoes.