Puzzles, Games and Strategies

1. (PCMI training) Im going to give each of you a hat to wear that is either black or white. You cannot see a hat on your head, but you can see a hat on anyone elses head. Your Goal as a group is to organize yourselves in a line, with all the white-hatted folks on one end, and all the black-hatted folks on the other end. Note that after I give you your hats, you cannot communicate with (nor touch) anyone. However, before I give you the hats you may jointly decide on a strategy. (a) Devise a strategy that will achieve the Goal. (b) Can you find a symmetric strategy, *i.e.*, one that is the same for all players?

2. (Internet Math puzzle, rumor: this is an interview problem for Microsoft) An airplane will make an around the world flight along the equator. It can carry fuel enough to fly half of entire trip. Other (same kind of) airplanes can help this airplane by carrying fuel and refueling other planes in air. We assume the following:

   (a) Each airplane must depart and return to the same airport, and that is the only airport they can land and refuel on ground.

   (b) Each airplane must have enough fuel to return to airport.

   (c) The time and fuel consumption of refueling can be ignored. (so we can also assume that one airplane can refuel more than one airplanes in air at the same time.)

   (d) The amount of fuel airplanes carrying can be zero as long as the other airplane is refueling these airplanes.

   What is the fewest number of airplanes and number of tanks of fuel needed to accomplish this work? (we only need airplane to go around the world)

3. (VT 1987) On Halloween, a black cat and a witch encounter each other near a large mirror positioned along the y-axis. The witch is invisible except by reflection in the mirror. At $t = 0$, the cat is at $(10, 10)$ and the witch is at $(10, 0)$. For $t > 0$, the witch moves toward the cat at a speed numerically equal to their distance of separation and the cat moves toward the apparent position of the witch, as seen by reflection, at a speed numerically equal to their reflected distance of separation. Denote by (u(t), v(t)) the position of the cat and by (x(t), y(t)) the position of the witch. (a) Set up the equations of motion of the cat and the witch for $t \geq 0$. (b) Solve for x(t) and u(t) and find the time when the cat strikes the mirror. (Recall that the mirror is a perpendicular bisector of the line joining an object with its apparent position as seen by reflection.)

4. (VT 1997) A business man works in New York and Los Angeles. If he is in New York, each day he has four options; to remain in New York, or to fly to Los Angeles by either the 8:00 a.m., 1:00 p.m. or 6:00 p.m. flight. On the other hand if he is in Los Angeles, he has only two options; to remain in Los Angeles, or to fly to New York by the 8:00 a.m. flight. In a 100 day period he has to be in New York both at the beginning of the first day of the period, and at the end of the last day of the period. How many different possible itineraries does the business man have for the 100 day period (for example if it was for a 2 day period rather than a 100 day period, the answer would be 4)?

5. (VT 1991) A and B play the following money game, where a_n and b_n denote the amount of holdings of A and B, respectively, after the n-th round. At each round a player pays one-half his holdings to the bank, then receives one dollar from the bank if the other player had less than c dollars at the end of the previous round. If $a_0 = .5$ and $b_0 = 0$, describe the behavior of a_n and b_n when n is large, for (i) $c = 1.24$ and (ii) $c = 1.26$.

6. (Internet math puzzle) Sally and Sue have a strong desire to date Sam. They all live on the same street yet neither Sally or Sue know where Sam lives. The houses on this street are numbered 1 to 99.

   Sally asks Sam "Is your house number a perfect square?". He answers. Then Sally asks "Is is greater than 50?". He answers again.

   Sally thinks she now knows the address of Sam's house and decides to visit.

   When she gets there, she finds out she is wrong. This is not surprising, considering Sam answered only the second question truthfully.

   Sue, unaware of Sally's conversation, asks Sam two questions. Sue asks "Is your house number a perfect cube?". He answers. She then asks "Is it greater than 25?". He answers again.

   Sue thinks she knows where Sam lives and decides to pay him a visit. She too is mistaken as Sam once again answered only the second question truthfully.

   If I tell you that Sam's number is less than Sue's or Sally's, and that the sum of their numbers is a perfect square multiplied by two, you should be able to figure out where all three of them live.

7. (Internet math puzzle) Mr. S. and Mr. P. are both perfect logicians, being able to correctly deduce any truth from any set of axioms. Two integers (not necessarily unique) are somehow chosen such that each is within some specified range. Mr. S. is given the sum of these two integers; Mr. P. is given the product of these two integers. After receiving these numbers, the two logicians do not have any communication at all except the following dialogue:

   (a) Mr. P.: I do not know the two numbers.
   (b) Mr. S.: I knew that you didn't know the two numbers.
   (c) Mr. P.: Now I know the two numbers.
   (d) Mr. S.: Now I know the two numbers.

   Given that the above statements are absolutely truthful, what are the two numbers?