Math 410 practice problem 2

Due: Nov. 27 (Monday) 10am to Jones 122 (Prof. Shi). Each problem is 10 points.
Solve as many problems as you can, but you should at least (attempt) give solutions of 6 problems. Try not to use any reference books, but if necessary check for any formulas important for you to solve the problems.

1. Determine all positive integers n for which $[2\sqrt{n}] = 1 + [\sqrt{n-1} + \sqrt{n+1}]$. Here $[n]$ is the greatest integer $\leq n$.

2. A fair die bearing number 1 through 6 on its faces is thrown repeatedly until the running total first exceeds 12. What is the most likely total that will be obtained?

3. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

4. Which is the larger of the two numbers

$$9^{9^{9^{\cdots 9}}}, (10\ 9\text{s}),\ \text{and}\ 10^{10^{10^{\cdots 10}}}, (9\ 10\text{s})?$$

5. Linda is staying with her very rich Uncle Bruce and even richer Aunt Joylene. Uncle Bruce loves dollar coins and 10 cent pieces but hates 20 cent and 50 cent coins, of which he has a vast supply. He tells Linda he will give her two 20 cent coins and one 50 cent coin for every dollar coin she gives him, and for every 10 cent coin he will give her a 50 cent coin. Aunt Joylene prefers 20 and 50 cent coins to 10 cent coins and dollars. She says she will give Linda a dollar coin and two 10 cents for every 50 cent coin, and a dollar coin for every 20 cent coin. Linda amuses herself by first taking a single dollar coin and swapping it with Bruce, she then takes the proceeds to Joylene and swaps them for dollars and ten cent coins. Then she goes back to Bruce, and so on. After n visits to Joylene, how much money will she have?

6. Twelve delegates attend a conference at which they are seated around a circular table. Each delegate brings with him to the conference three copies of a document which he has prepared. He gives one copy to each of his two immediate neighbors at the table, and keeps the third copy himself. While the delegates are away at coffee, having left the papers in question on their chairs, a thief rushes into the room and snatches one document at random from each chair. What is the probability that the thief obtains a complete set of the twelve documents?

7. If $a + b + c = 0$, prove that

$$\frac{a^5 + b^5 + c^5}{5} = \frac{a^3 + b^3 + c^3}{3} \cdot \frac{a^2 + b^2 + c^2}{2}$$

use simplest possible arguments.

8. Prove
$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n+1} n \binom{n}{n} = \begin{cases} 0 & if \;\; n > 1, \\ 1 & if \;\; n = 1. \end{cases}$$

9. The lock on a safe consists of three wheels A, B, and C, each of which may be set in eight different positions. Due to a defect in the mechanism, the door will open when any two of the wheels are in the correct position. Thus anybody can open the safe in 64 tries by letting B run through all 8 position for *each* position of A. However, the safe can be always opened in far fewer tries than that. What is the minimum number of tries that is guaranteed to open the safe?

10. P and Q are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$, and $I - (P + Q)$ is invertible. Show that P and Q have the same rank.