Midterm practice

Due: Oct. 20 (Friday) 5pm to Jones 122 (Prof. Shi). Each problem is 10 points.
Solve as many problems as you can, but you should at least (attempt) give solutions of 6 problems. Try not to use any reference books, but if necessary check for any formulas important for you to solve the problems.

1. The Fibonacci sequence is given by $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$. Prove that $\displaystyle\sum_{k=1}^{n} \frac{a_k}{2^k} < 2$.
2. (VT 1981) Define $F(x)$ by $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (wherever the series converges), where F_n is the n-th Fibonacci number defined by $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n > 1$. Find an explicit closed form (without summation) for $F(x)$.
3. (VT 1981) With k a positive integer, prove that $(1 - k^{-2})^k \geq 1 - 1/k$.
4. (VT 1979) Show, for all positive integers $n = 1, 2, ..$, that 14 divides $3^{4n+2} + 5^{2n+1}$.
5. Find the sum: $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{-[3m+n+(m+n)^2]}$.
6. Randomly select 55 numbers from the set $S = \{1, 2, 3, \cdots, 99, 100\}$. Prove
   (a) There exists two selected numbers whose difference is 10.
   (b) There exists two selected numbers whose difference is 12.
   (c) It is possible that no any two selected numbers whose difference is 11.
7. (VT 1980) Let $*$ denote a binary operation on a set S with the property that $(w*x)*(y*z) = w*z$ for all $w, x, y, z \in S$. Show (a) If $a*b = c$, then $c*c = c$. (b) If $a*b = c$, then $a*x = c*x$ for all $x \in S$.
8. (VT 1989) Three farmers sell chickens at a market. One has 10 chickens, another has 16, and the third has 26. Each farmer sells at least one, but not all, of his chickens before noon, all farmers selling at the same price per chicken. Later in the day each sells his remaining chickens, all again selling at the same reduced price. If each farmer received a total of \$35 from the sale of his chickens, what was the selling price before noon and the selling price after noon?
9. (VT 1988) A man goes into a bank to cash a check. The teller mistakenly reverses the amounts and gives the man cents for dollars and dollars for cents. (Example: if the check was for \$5.10, the man was given \$10.05.). After spending five cents, the man finds that he still has twice as much as the original check amount. What was the original check amount? Find all possible solutions.
10. Suppose that $p \geq 5$ is a prime, and $2p + 1$ is also a prime. Prove that $4p + 1$ is not a prime.
11. Each of ten line segments is longer than 1 but shorter than 55. Prove that you can select three line segments from these ten to form a triangle.
12. For any positive integer n, prove that there exists a multiple of 2^n whose each digit (base 10) is not zero.