Exact Multiplicity of Solutions and S-shaped Bifurcation Curve for a Class of Semi-linear Elliptic Equations from a Chemical Reaction Model

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二OO五年四月
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摘 要

分歧理论是近三十年来逐步形成的有重要应用价值的数学分支, 其应用主要体现在生物数学、经济数学、管理数学等学科。一般说来, 分歧图象扮演着相当重要的角色。因此, 讨论一类具体问题的分歧图象和解的个数就成了近年来一个热门的问题。本文共分为四部分: 

第一章介绍了分歧理论中一些具体的半线性椭圆方程 Δu + λf(u) = 0 问题的近几十年研究成果。

第二章从一个自催化化学反应中建立了一个数学模型, 进而提出了方程

\[
\begin{aligned}
\Delta u + \lambda f(u) &= 0 & & x \in \Omega, \\
u(x) &= 0 & & x \in \partial \Omega.
\end{aligned}
\]

正解的个数及其分歧图的问题, 其中 \( f(u) = u^p - u^{p+1} \), \( p > 1 \), \( \Omega \) 为 \( R^n \) 中 1 或 2 维单位球。

第三章运用连续方法主要研究了第二章所提出方程的正解个数, 并得到了它的分歧图象。

本文的最主要结果将在第四章得到, 因为可以用扰动的眼光看待方程

\[
\begin{aligned}
\Delta u + \lambda f(u + k) &= 0 & & x \in \Omega, \\
u(x) &= 0 & & x \in \partial \Omega.
\end{aligned}
\]

其中, \( k > 0 \) 为常数。那么利用第三章结论, 适当压缩 \( k \) 的值, 综合讨论解曲线在 “退化点” 处的性质, 我们得到了分歧解曲线。

关 键 词: 最大值原理, 退化解, 分歧, 局部解曲线, S 形曲线。
Abstract

Bifurcation Theory is gradually becoming a valuable branch of mathematics in recent three or four decades. Its theories and applications are represented in lots of subjects, e.g. Biological Mathematics, Economy Mathematics, Management Mathematics, and so on. Generally speaking, bifurcation diagram serves as a very important role in Bifurcation Theory. However, the diversity of nonlinearity $f$ decides the various forms of the corresponding bifurcation diagram, so do the researching tools on the problem.

In this paper, we get an exact multiplicity of positive solutions and the explicit bifurcation diagram on a class of semi-linear elliptic equation. In Chapter 1, introductions about several kinds of semi-linear elliptic equations and their recent results. From an autocatalytic chemical reaction in Chapter 2, we established a mathematical model. Furthermore, we propose the problem of positive solutions on the equation

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(u) = u^p - u^{p+1}$, $p > 1$, $\Omega$ is an unit ball in $\mathbb{R}^n$, $n = 1$ or 2. Of course, the research on bifurcation diagram attract us most. And in Chapter 3, taking advantage of the ”continuation method”, we study the multiplicity of positive solutions to the equation in Chapter 2, and the explicit bifurcation curve is obtained. The main or most important result of us will be got till Chapter 4. Since we can regard the equation

$$\begin{cases} \Delta u + \lambda f(u + k) = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

as a perturbed one as in Chapter 3, where $k > 0$ is a constant. Then we use the fruits in Chapter 3, compress the value of $k$ appropriately, and discuss the properties of the ”degenerate point” on solution curve, to achieve the proof of the result that the bifurcation diagram is S-shaped.

Keywords: Maximum Principle, degenerate solution, bifurcation, local solution curve, S-shaped curve.
Chapter 1  Introduction

In general, bifurcation phenomena occur frequently in solving nonlinear equations. If one or more parameters in the equations vary, then the qualitative behavior of the system may change. Such a change is called a bifurcation. If the equation has the form

\[ F(\lambda, u) = 0, \]  \hspace{1cm} (1.1)

where \( \lambda \) is a parameter, then the number of solutions in the above equation may change when parameter \( \lambda \) changes. We call the number of solutions multiplicity in the equation. Bifurcation Theory attempts to explain various phenomena that have been discovered and described in the natural science over the centuries, and it has become an important branch nonlinear analysis. Hansjörg Kielhöfer gives a basic idea of Bifurcation Theory and some applications to PDEs in [Ha].

In recent years, extensive works have been done on the bifurcation of solutions to semi-linear equation of the type:

\[ \Delta u + \lambda f(u) = 0, \text{ in } \Omega, \; u = 0 \text{ on } \partial \Omega \]  \hspace{1cm} (1.2)

where \( \Omega \) is a bounded smooth domain, and \( \lambda \) is a parameter. Because of the diversity of nonlinearity \( f \), one will get kinds of results through different ways or methods and for various explicit nonlinearities \( f' \)s. In 1980’s, P.L. lions conjectured that the structure of solution set \( \{ (\lambda, u) \} \) of (1.2) is similar to the structure of the solution set \( \{ (\lambda, u) \} \) of algebraic equation \( \lambda_1 u = \lambda f(u) \), where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) on \( H_0^1(\Omega) \). As Lions [PL] pointed out, it was only a formal way of guessing the bifurcation diagram at that time. Generally speaking, the bifurcation diagram can be much more complicated if the domain \( \Omega \) is complicated or the nonlinearity \( f \) grows super-critically. When \( \Omega \) is a unit ball in \( R^n \), the bifurcation diagram and the exact multiplicity of solutions can be determined by the properties of nonlineariries \( f' \)s. T. Ouyang and J. Shi [OS1] established an exact multiplicity result of positive solutions of (1.2) when \( f \) satisfies that \( f'' \) changes sign only once and asymptotically sub-linear or linear, where \( \Omega \) is a unit ball in \( R^n \). This class of nonlinearities include \( f(u) = u(u - b)(c - u) \) for \( 0 < 2b < c \) and \( f(u) = u^p - u^q \) for \( 1 < p < q \), where \( b, c, p, q \) are constants.
Furthermore, they [OS2] showed that the shape of the bifurcation curve depends on the shape of the graph of function $f(u)/u$ as well as the growth rate of $f$.

When $f(u) = e^{u/(1+\varepsilon u)}$, $\varepsilon > 0$ is a constant, problem (1.2) become the so-called perturbed Gelfand Equation (see [BE] for more backgrounds of this problem)

$$-\Delta v = \lambda e^{v/(1+\varepsilon v)} \text{ in } B^n, \ v = 0 \text{ on } \partial B^n,$$

where $B^n$ is an unit ball in $\mathbb{R}^n$. For $n = 1$ or 2, it was conjectured and supported by numerical evidences that the solution curve is exactly $S$-shaped when $\varepsilon$ is sufficiently small. In 1974, S.V.Parter [Pa] proved that for $n = 2$ and any $\varepsilon > 0$ small, there exists $\lambda_1(\varepsilon) < \lambda_1(\varepsilon) < \lambda_2(\varepsilon)$ such that (1.2) has a unique positive solution if $\lambda \in (0, \lambda_1(\varepsilon)) \cup (\lambda_2(\varepsilon), +\infty)$, and (1.2) has at least three positive solutions if $\lambda \in [\lambda_1(\varepsilon), \lambda_2(\varepsilon)]$. In 1980, E.N.Dancer [Da] proved that for any small positive $\lambda_0 > 0$ and $n = 1$ or 2, one can find an $\varepsilon_0 > 0$ small such that if $\varepsilon \in (0, \varepsilon_0)$ then there is a constant $\lambda_2(\varepsilon) > 0$ such that (1.2) has exactly three positive solutions if $\lambda \in (\lambda_0, \lambda_2(\varepsilon))$, exactly two positive solutions if $\lambda = \lambda_2(\varepsilon)$ and there is an unique positive solution if $\lambda > \lambda_2(\varepsilon)$. But for small $\lambda$-range: $0 < \lambda < \lambda_0$, the conjecture remained unsolved. In 1985, S.P.Hastings and J.B.McLeod [HM] taking used to prove that the conjecture is true for the case $n = 1$. In 1994, S.-H.Wang [Wa] further proved, also by quadratures, that when $n = 1$, and $\varepsilon \leq 1/4.4967$, the conjecture holds. In [KL], the upper bound for $\varepsilon$ was improved to $1/4.35$ by P. Korman and Y. Li, where they use local bifurcation arguments and the developed techniques in [KLO1], together with quadratures. For $n = 2$, Y. Du and Y. Lou [DL] proved that the perturbed Gelfand Equation (1.3) has an exactly $S$-shaped bifurcation curve when $\varepsilon$ is small enough. It is well-known that the conjecture can not be extended to $n \geq 3$ as results in [Da] showed that (1.3) can have a large number of solutions for certain values of $\lambda$ when $3 \leq n \leq 9$.

When $f(u) = \lambda(1 + \varepsilon u)^m e^{u/(1+\varepsilon u)}$, $\varepsilon > 0$ in (1.2), Y. Du [Du] rigorously confirmed (1.2) has an $S$-shaped solution curve when $\varepsilon$ is small enough for $n = 1$ or 2, by making use of a new limiting equation and a continuation method based on a local bifurcation theorem of Crandall and Rabinowitz. The bifurcation analysis near turning points was first developed by P. Korman, Y. Li and T. Ouyang [KLO1, KLO2] for $n = 1$ or 2, provided that $f$ being a generalized cubic
function, and they gave a new proof to the result of [SW], which is for \( n = 1 \).

In this paper, we focus on the problem

\[
\Delta u + \lambda f(u + k) = 0, \text{ in } B^n, \quad u = 0, \text{ on } \partial B^n, \quad (1.4)
\]

where \( f(u) = u^p - u^{p+1} \), \( B^n \) is the unit ball in \( \mathbb{R}^n \) \( (n \geq 1) \), \( k \geq 0 \), \( p \geq 1 \) are constants. Here \( \lambda \) is treated as a bifurcation parameter. By a positive solution, we mean \( u(x) > 0 \) for \( x \in \Omega \). Problem (1.4) arises from a model in applied chemistry.

We organize this paper as follows. In Chapter 2, we drive the problem from a mathematical model in Applied Chemistry. Then in Chapter 3, we get the exact multiplicity of positive solutions to the problem. And the main result will be proved in Chapter 4. After discussing the "turning point" and the number \( k > 0 \) small enough, we get the S-shaped solution curve.
Chapter 2  A Mathematical Model in Chemistry and Some Preliminaries

The prototype representation for an autocatalytic chemical reaction is

\[ A + pB \rightarrow (p + 1)B, \quad (2.1) \]

and the reaction rate is \( k a b^p \), where \( a \) and \( b \) are the concentrations of the reactant \( A \) and the autocatalyst \( B \), and \( p \geq 1 \) is the order of the reaction with respect to the autocatalytic species [GS]. The equations describing the reaction and diffusion of the two reactants \( A \) and \( B \) in a bounded region are

\[
\frac{\partial a}{\partial t} = D_A \Delta a - ab^p, \quad \frac{\partial b}{\partial t} = D_B \Delta b + ab^p, \quad t > 0, \quad x \in \Omega, \quad (2.2)
\]

where \( D_A \) and \( D_B \) are the diffusion coefficients of \( A \) and \( B \) respectively, and \( \Omega \) is a bounded reactor in \( \mathbb{R}^n \). Here the spatial dimension is \( 1 \leq n \leq 3 \), and the typical geometry of the reactor \( \Omega \) is spherical \((n = 3, \text{ and } \Omega = B^3)\), cylindrical \((n = 2, \text{ and } \Omega = B^2)\), and linear \((n = 1, \text{ and } \Omega = (-1, 1))\), where \( B^n = \{ x \in \mathbb{R}^n : |x| < 1 \} \) is the unit ball in \( \mathbb{R}^n \). The chemical \( A \) and \( B \) can diffuse from a reservoir of constant composition across the boundary \( \partial \Omega \) into \( \Omega \), thus the boundary conditions of \( A \) and \( B \) can be taken as

\[ a(x, t) = a_0 > 0, \text{ and } b(x, t) = b_0 \geq 0, \quad x \in \partial \Omega. \quad (2.3) \]

In the case of equal diffusion coefficients \( D_A = D_B = D \), the steady state solutions of (2.2) satisfy

\[
\begin{cases}
  D \Delta a - ab^p = 0, \quad D \Delta b + ab^p = 0, & x \in \Omega, \\
  a(x) = a_0, \quad b(x) = b_0, & x \in \partial \Omega.
\end{cases} \quad (2.4)
\]

By adding the two equations above, we have
\[
\begin{aligned}
\Delta (a(x) + b(x)) &= 0, \quad x \in \Omega, \\
(a(x) + b(x)) &= a_0 + b_0, \quad x \in \partial \Omega.
\end{aligned}
\]

**Lemma 2.1** If \(a(x), b(x) \in C^2(\Omega)\) satisfy (2.5) then

\[
a(x) + b(x) \equiv a_0 + b_0
\]

for any \(x \in \Omega\), where \(a_0 > 0, b_0 \geq 0\) are constants.

**Proof.** Define

\[
u(t, x) = \begin{cases}
a(x) + b(x), & (t, x) \in (0, T] \times \bar{\Omega} \\
a_0 + b_0, & (t, x) \in \{t = 0\} \times \bar{\Omega},
\end{cases}
\]

then we have

\[
\begin{aligned}
\frac{\partial u(x)}{\partial t} + \Delta u(x) &= 0, \quad (t, x) \in (0, T] \times \Omega, \\
u(t, x) &= a_0 + b_0, \quad (t, x) \in (0, T] \times \partial \Omega, \\
u(0, x) &= a_0 + b_0, \quad x \in \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial v(x)}{\partial t} + \Delta v(x) &= 0, \quad (t, x) \in (0, T] \times \Omega, \\
v(t, x) &= a_0 + b_0, \quad (t, x) \in (0, T] \times \partial \Omega, \\
v(0, x) &= a_0 + b_0, \quad x \in \Omega
\end{aligned}
\]

By the uniqueness of the solution of Heat Equation, we have \(u(t, x) \equiv v(t, x), \quad (t, x) \in [0, T] \times \Omega\). Hence

\[
a(x) + b(x) = a_0 + b_0, \quad x \in \Omega.
\]

\[\diamondsuit\]

By Lemma 2.1, the system (2.4) can be reduced one equation

\[
\begin{aligned}
\Delta (a(x) + b(x)) &= 0, \quad x \in \Omega, \\
(a(x) + b(x)) &= a_0 + b_0, \quad x \in \partial \Omega.
\end{aligned}
\]

(2.5)

Now take \(v(x) = \frac{b(x)}{a_0 + b_0}, \quad \lambda = D^{-1}(a_0 + b_0)^p\), then \(v(x)\) satisfies
\[
\begin{aligned}
\begin{cases}
\Delta v + \lambda (1 - v(x))(v(x))^p = 0, & x \in \Omega, \\
v(x) = k, & x \in \partial \Omega
\end{cases}
\end{aligned}
\] (2.7)

where \( k = \frac{b_0}{a_0 + b_0} \). So we obtain another result:

**Lemma 2.2** If \( v(x) \in C^2(\overline{\Omega}) \) is a positive solution of (2.7), then \( k \leq v(x) \leq 1 \), where \( k \in [0, 1) \), \( \lambda > 0 \).

**Proof.** (i) Let \( M = \max_{x \in \overline{\Omega}} v(x) \). Suppose \( M > 1 \), then there must exist some \( x^* \in \Omega \) such that \( M = v(x^*) > 1 \). Thus \( \nabla v(x^*) = 0 \), and \( \Delta v(x^*) \leq 0 \). Note that \( v(x) > 0 \), so we obtain

\[
\lambda (v(x^*) - 1)(v(x^*))^p > 0.
\]

Again by (2.7) we have

\[
0 \geq \Delta v(x^*) = \lambda (v(x^*) - 1)(v(x^*))^p > 0,
\]

a contradiction. Hence \( v(x) \leq 1 \) for any \( x \in \overline{\Omega} \).

(ii) Since \( \Delta v(x) = \lambda (v(x) - 1)(v(x))^p \leq 0 \) for any \( x \in \Omega \), we have

\[
\begin{aligned}
\begin{cases}
-\Delta v(x) \geq 0, & x \in \Omega, \\
v(x) = k, & x \in \partial \Omega,
\end{cases}
\end{aligned}
\]

i.e. \( v(x) \) is the sup-solution of the boundary value problem

\[
\begin{aligned}
\begin{cases}
-\Delta v(x) = 0, & x \in \Omega, \\
v(x) = k, & x \in \Omega.
\end{cases}
\end{aligned}
\]

From the weak maximum principle (see [LE], Chapter 6, Theorem 2), we have \( v(x) \geq k \), for any \( x \in \overline{\Omega} \).

**Definition 2.1** Let \( \Omega \) be the domain of function \( u(x) \). If \( u(x) \) satisfies (1.1) for any \( x \in \Omega \), then we call \( u(x) \) or \( (\lambda, u) \) is a solution of (1.1), and \( \lambda \) is called a bifurcation parameter.

**Definition 2.2** If \( (\lambda, u) \) is a solution of (1.1). Then the orthogonal coordinate system, of which \( \lambda - \) axis and \( u(x_0) = u \) axis consist for \( x_0 \in \Omega \), is called a phase plane; the orthogonal coordinate system, of which \( \lambda - \) axis and \( u - \) axis consist, is called a phase space. And \( (\lambda, u) \) is called a point in phase space.
Definition 2.3 Let $\Gamma$ be a solution curve in phase space, i.e. any point $(\lambda, u)$ on $\Gamma$ is a solution of (1.1). Suppose there is a point $(\lambda_0, u_0)$ on $\Gamma$, satisfying the number of solutions(multiplicity) changes as $\lambda$ crosses $\lambda_0$. Then $(\lambda_0, u_0)$ is called a bifurcation solution or bifurcation point in phase space, and $\Gamma$ is called a bifurcation curve.

Definition 2.4 If $\Gamma$ is a bifurcation curve, then the diagram where $\Gamma$ behaves is called a bifurcation diagram.
Chapter 3  Positive Solutions and The Bifurcation Diagram

In the following, Ω is the unit ball, let \( k \geq 0, \ p \geq 1 \), we consider the problem

\[
\begin{cases}
\Delta u(x) + \lambda (1 - k - u(x))(k + u(x))^p = 0, & x \in \Omega, \\
0 \leq u(x) \leq 1 - k, & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega. 
\end{cases}
\]  

(3.1)

This suggests that

\[
\begin{cases}
\Delta v(x) + \eta (1 - v(x))(v(x))^p = 0, & x \in \Omega, \\
0 \leq v(x) \leq 1, & x \in \Omega, \\
v(x) = 0, & x \in \partial \Omega. 
\end{cases}
\]  

(3.2)

is a good approximation of (3.1), when \( k \) is small.

Let \( f(v) = (1 - v)v^p \), then (3.1) become

\[
\begin{cases}
\Delta u(x) + \lambda f(u(x) + k) = 0, & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega. 
\end{cases}
\]  

(3.3)

and (3.2) becomes

\[
\begin{cases}
\Delta v(x) + \eta f(v(x)) = 0, & x \in \Omega, \\
v(x) = 0, & x \in \partial \Omega. 
\end{cases}
\]  

(3.4)

The linearized equation of (3.3) is

\[
\begin{cases}
\Delta w(x) + \lambda f'(u(x) + k)w(x) = 0, & x \in \Omega, \\
w(x) = 0, & x \in \partial \Omega. 
\end{cases}
\]  

(3.5)

and the linearized equation of (3.4) is

\[
\begin{cases}
\Delta \phi + \eta f'(v(x))\phi = 0, & x \in \Omega, \\
\phi(x) = 0, & x \in \partial \Omega. 
\end{cases}
\]  

(3.6)

We have a lemma:
**Lemma 3.1** \( p \geq 1, 0 \leq k < 1 \). Let \( u(x), v(x) \) be the solutions of (3.3) and (3.5) respectively, \( w(x) > 0 \) for all \( x \in \Omega \), then

\[
\int_{\Omega} f(u(x) + k)w(x)dx > 0.
\]

**Proof.** By Lemma 2.3 in [OS1], we have

\[
\int_{\Omega} f(u(x) + k)w(x)dx = \frac{1}{2\lambda} \int_{\partial\Omega} |\nabla u(x)| \cdot |\nabla w(x)|(x \cdot \nu)ds,
\]

where \( \nu \) is the outer unit normal vector to \( \partial\Omega \). Since \( \Omega \subset \mathbb{R}^n \) is the unit ball, then \( x \cdot \nu \geq 0 \) for all \( x \in \partial\Omega \), and \( \text{mes}\{x \in \partial\Omega : x \cdot \nu > 0\} > 0 \). \( w(x) \) is a nontrivial solution of (3.5), then \( |\nabla w(x)| > 0 \). Note that \( 0 \leq u(x) \leq 1 - k \), \( x \in \Omega \), we have

\[
\begin{cases}
-\Delta u(x) = \lambda (1 - k - u(x))(k + u(x))^p \geq 0, & \text{in } \Omega, \\
u u(x) = 0, & \text{on } \partial\Omega.
\end{cases}
\]

By strong maximum principle, we obtain \( u(x) > 0 = u(x)|_{\partial\Omega} \) for all \( x \in \Omega \). It follows from Hopf’s Lemma (see [LE], page 330, Lemma) that

\[
\frac{\partial u(x)}{\partial \nu} |_{\partial\Omega} < 0.
\]

Hence \( |\nabla u(x)| > 0 \) for \( x \in \Omega \), furthermore we obtain

\[
\int_{\Omega} f(u(x) + k)w(x)dx > 0.
\]

\( \checkmark \)

Our first result is:

**Theorem 3.1** Let \( k > 0 \) be small, \( u(x) \) be a degenerate solution of (3.3), and \( w(x) \) be the nontrivial positive solution of (3.5). If \( n = 1 \) or \( 2 \), then \( w(x) \) does not change sign in \( \Omega \).

**Proof.** By a well-known result in [GNN], \( u(x) \) is radially symmetric: \( u(x) = u(r) \), \( r = |x| \), \( u'(r) < 0 \) for any \( r \in (0, 1] \), and satisfying

\[
\begin{cases}
u u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u(r) + k) = 0 & r \in (0, 1) \\
u'(0) = u(1) = 0.
\end{cases}
\]

From Proposition 3.3 in [LN], \( w(x) \) is also radially symmetric:
\[
\begin{cases}
w''(r) + \frac{n-1}{r}w'(r) + \lambda f'(u(r) + k)w(x) = 0 & r \in (0, 1) \\
w'(0) = w(1) = 0,
\end{cases}
\]

it follows that \(w(x) \neq 0\) (see Lemma 3.1 in [HK]). We may assume that \(w(0) > 0\).

If \(k = 0\), then by [OS1] we have \(w(r) > 0\) for \(r \in (0, 1)\). Now \(k > 0\), we make use of the test function \(v(r) = ru'(r) + \beta\), where \(\beta\) is a positive constant to be determined later. By a straight calculation: \(v' = u'+ ru'', v'' = 2u'' + ru''\). Set

\[
G(r) = v'' + \frac{n-1}{r}v' + \lambda f'(u + k)v
\]

\[
= (2u'' + ru'') + \frac{n-1}{r}(u' + ru'') + \lambda f'(u + k)(ru' + \beta)
\]

\[
= 2u'' + ru'' + \frac{n-1}{r}u' + \frac{(n-1)r}{r}u'' + \lambda rf'(u + k)u' + \lambda \beta f'(u + k),
\]

note that

\[
\frac{d}{dr} \left[ u'' + \frac{n-1}{r}u' + \lambda f(u + k) \right] = u''' + \left( \frac{n-1}{r}u'' - \frac{n-1}{r^2}u' \right) + \lambda f'(u + k)u'
\]

\[
= 0.
\]

Hence

\[
G(r) = 2u'' + \frac{2(n-1)}{r}u' + \lambda \beta f'(u + k)
\]

\[
= 2 \left[ u'' + \frac{n-1}{r}u' + \lambda f(u + k) \right] - 2\lambda f(u + k) + \lambda \beta f'(u + k)
\]

\[
= \lambda |\beta f'(u + k) - 2f(u + k)|
\]

\[
= \lambda f(u + k)g(r),
\]

where

\[
g(r) = \frac{\beta f'(u + k)}{f(u + k)} - 2 = \beta \left[ \frac{p}{u + k} + \frac{1}{u + k - 1} \right] - 2.
\]

Then we claim: \([r^{n-1}(v'w - vw')]' = G(r)r^{n-1}w \quad (\ast)\), \(G(r) \equiv \lambda f(u + k)g(r)\).
Proof of the claim:

\[ [r^{n-1}(v'w - vw')]' = (n-1)r^{n-2}[v'w - vw'] 
+ r^{n-1}[(v''w + v'w') - (v'w' + vw'')] 
= r^{n-2}[(n-1)(v'w - vw') + r(v''w - vw'')] 
= r^{n-2}[(n-1)v'w - (n-1)vw' + rv''w - rvw''] 
= r^{n-2}[rw(v'' + \frac{n-1}{r}v') - rv(w'' + \frac{n-1}{r}w')] 
= r^{n-2}[rw(G(r) - \lambda f'(u + k)v) - rv(-\lambda f'(u + k)w)] 
= r^{n-1}G(r)w. \]

Clearly, \( g(r) \) is increasing in \( r \), since \( u(r) \) is decreasing in \( r \). Now suppose \( w(r) \) changes sign in \((0, 1)\) and want to deduce a contradiction from this. Let \( r_0 \in (0, 1) \) be the first root of \( w(r) = 0 : w(r_0) = 0 \), and \( w(r) > 0 \) for \( r \in [0, r_0) \).
We take \( \beta = -r_0u'(r_0) \). Since

\[ v' = -r\lambda f(u + k) + (2 - n)u' < 0 \]

for any \( r \in (0, 1] \) when \( n = 1 \) or 2, we have \( v(r) > v(r_0) = 0 \) on \([0, r_0)\) and \( v(r) < 0 \) on \((r_0, 1]\). We will find a contradiction through the two cases below by quadratures.

Case (i): \( g(r) \leq 0 \). So we have \( g(r) < g(r_0) \leq 0 \) on \([0, r_0)\) at this case. By integrating (*) from 0 to \( r_0 \), we will obtain

\[ 0 > \int_0^{r_0} r^{n-1}\lambda f(u + k)g(r)w(r)dr = \int_0^{r_0} r^{n-1}G(r)w(r)dr 
= [r^{n-1}(v'w - vw')]_0^{r_0} 
= r_0^{n-1}[v'(r_0)w(r_0) - v(r_0)w'(r_0)] 
= 0, \]

since \( w(r_0) = v(r_0) = 0 \). Obviously, it is a contradiction.

Case (ii): \( g(r) > 0 \). At this time we consider the last root \( r^0 \) of \( w(r) = 0 \) before \( r = 1 \): \( r_0 \leq r^0 < 1 \), \( w(r^0) = 0 \), \( w(r) \neq 0 \) on \((r^0, 1)\). We may assume \( w(r) > 0 \) (otherwise we take \(-w(r)\)) on \((r^0, 1)\), then \( w'(r^0) > 0 > w'(1) \) since \( w(r^0) = w(1) = 0 \). Now using \( g(r) > 0 \) and \( v(r) \leq 0 \) on \([r^0, 1]\) and integrating
from \( r^0 \) to 1, we will obtain
\[
0 \leq \int_{r^0}^1 r^{n-1} \lambda f(u + k) g(r) w(r) dr = \int_{r^0}^1 G(r) r^{n-1} w(r) dr
\]
\[
= [r^{n-1} (v'w - vw')]\big|_{r^0}^{1}
\]
\[
= [v'(1) w(1) - v(1) w'(1)] - (r^0)^{n-1} [v'(r^0) w(r^0) - v(r^0) w'(r^0)]
\]
\[
= (r^0)^{n-1} v(r^0) w'(r^0) - v(1) w'(1) \leq 0,
\]
since \( v(r^0) < 0, v(1) < 0, w(1) = w(r^0) = 0 \). It is another contradiction. So we finish the proof.

Next we will illustrate the parameter \( \eta \) of any positive solution to (3.4) is larger than zero by a lemma:

**Lemma 3.2** If \( n \leq 2 \), \( 1 < p < \infty \); or \( n \geq 3 \), \( 1 < p < \frac{4}{n+2} \). Then

(i) There exists an \( \eta_1 > 0 \) such that (3.4) has at least a positive solution if \( \eta \geq \eta_1 \);

(ii) the set
\[
A = \{(\eta, v(\cdot, \eta)) \in \mathbb{R} \times C^2(\Omega) \mid v(\cdot, \eta) \text{ is the positive solution of (3.4), } \eta \geq \eta_1 \}
\]
is one to one with the set
\[
B = \{(\eta, v(0, \eta)) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid v(\cdot, \eta) \text{ is the positive solution of (3.4), } \eta \geq \eta_1 \}
\]

**Proof.** (i). The proof is similar to the one of THEOREM 2.32 on page 12 to 14 in [R].

(ii). Define \( T : A \longrightarrow B \) by \( v(0, \eta) = Tv(\cdot, \eta), \, \eta \geq \eta_1 \). We only need to prove that \( T \) is injective.

Suppose that
\[
Tv_0(\cdot, \eta_0) = Tv(\cdot, \eta),
\]
i.e.
\[
v_0(0, \eta_0) = v(0, \eta).
\]
We must prove that
\[
\eta = \eta_0 \quad \text{and} \quad v = v_0.
\]
By [GNN], both $v_0$ and $v$ are radially symmetric. Now let $u(r) = v((\eta_0/\eta)^{1/2}r)$, we have

$$u'(r) = v'((\eta_0/\eta)^{1/2}r)(\eta_0/\eta)^{1/2} \quad \text{and} \quad u''(r) = v''((\eta_0/\eta)^{1/2}r)(\eta_0/\eta).$$

So

$$u''(r) + \frac{n-1}{r} u'(r) + \eta_0 f(u(r))$$

$$= v''((\eta_0/\eta)^{1/2}r)(\eta_0/\eta) + \frac{n-1}{r} v'((\eta_0/\eta)^{1/2}r)(\eta_0/\eta)^{1/2} + \eta_0 f(v((\eta_0/\eta)^{1/2}r))$$

$$= (\eta_0/\eta) [v''(s) + \frac{n-1}{s} v'(s) + \eta f(v(s))]$$

$$= 0,$$

where $s = (\eta_0/\eta)^{1/2}r$. Note that $u(0) = v(0) = v(0, \eta)$ and $u'(0) = 0$, i.e.

$$\begin{cases}
    u''(r) + \frac{n-1}{r} u'(r) + \eta_0 f(u(r)) = 0, & \text{in } (0, 1], \\
    u'(0) = 0, u(0) = v(0, \eta) = v_0(0, \eta_0).
\end{cases}$$

Since $v_0$ satisfies the above equation with the same initial value, by uniqueness of solution to the above initial value problem, we deduce $u(r) = v_0(r) > 0$ for $r \in [0, 1)$ and $u(1) = v_0(1) = 0$. This implies that $\eta = \eta_0$, and hence $u(r) = v(r) = v_0(r)$, i.e.

$$v(\cdot, \eta) = v_0(\cdot, \eta_0).$$

Lemma 3.3 Suppose that $v_0$ is a degenerate positive solution of (3.4) with $\eta = \eta_0 > 0$. Then all solutions near $(\eta_0, v_0)$ lie on a smooth curve represented by $(\eta_0 + \tau(s), v_0 + O(s))$ with $s$ small, where $\tau(0) = \tau'(0) = 0$ and $\tau''(0) > 0$.

Proof. Let $F : R_+ \times X \rightarrow Y$ be defined by

$$F(\eta, v) = \Delta v + \eta f(v),$$

where $X = C_0^{2, \alpha}(\bar{\Omega})$, $Y = C^{\alpha}(\bar{\Omega})$. It is easy to see that $F$ is a smooth mapping from $R_+ \times X$ to $Y$, and

$$\langle F_v(\eta, v), w \rangle = \Delta w + \eta f'(v)w,$$

where $f(v) = v^p - v^{p+1}$. 

\end{document}
The null space of $F_v(\eta_0, v_0)$ is one dimensional, and $\text{codim} R(F_v(\eta_0, v_0)) = 1$ since $F_v(\eta_0, v_0)$ is a Fredholm operator of index 0. Now let

$$N(F_v(\eta_0, v_0)) = \text{span}\{\phi_0\}, \quad \phi_0(x) \geq 0 \text{ for } x \in \Omega,$$

then

$$\begin{cases} 
\Delta \phi_0 + \eta_0 f'(v_0)\phi_0 = 0, & \text{in } \Omega, \\
\phi_0 = 0, & \text{on } \partial \Omega.
\end{cases}$$

Moreover,

$$F_v(\eta_0, v_0) = f(v_0) \in R(F_v(\eta_0, v_0)).$$

If this is not true, one can find some $z \in X$ such that

$$\begin{cases} 
\Delta z + \eta_0 f'(v_0)z = f(v_0), & \text{in } \Omega, \\
z = 0, & \text{on } \partial \Omega.
\end{cases}$$

So we have

$$\int_{\Omega} f(v_0)\phi_0 \, dx = \int_{\Omega} [\Delta z\phi_0 + \eta_0 f'(v_0)z\phi_0] \, dx = \int_{\Omega} [\Delta \phi_0 + \eta_0 f'(v_0)\phi_0] \, dzx = 0,$$

which contradicts Lemma 3.1. Now applying Bifurcation Theorem of Crandall and Rabinnowicz ([CR]), we can conclude that $(\eta_0, v_0)$ is a bifurcation point, and near $(\eta_0, v_0)$ the solutions of (3.4) form a curve $(\eta_0 + \tau(s), \eta_0 + s\phi_0 + x(s))$ with $s$ near 0 and $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$. From the Implicit Function Theorem we know that the solution curve near $(\eta_0, v_0)$ is smooth.

For the sign of $\tau''(0)$, we first claim that

$$(i) \quad \tau''(0) = -\eta_0 \int_{\Omega} f''(v_0)\phi_0^2 \, dx = -\int_{\Omega} \frac{\eta_0 \int_0^1 r^{n-1} f''(v_0)\phi_0^2 dr}{\int_0^1 f(v_0)\phi_0^2 dr}$$

(see [OS1], page 133). Recall Lemma 3.1, one can get $\int_0^1 r^{n-1} f(v_0)\phi_0^2 dr > 0$. Then we claim that

$$\int_0^1 r^{n-1} f''(v_0)[v_{br}]^2 \phi_0 dr = 0.$$
Proof of the claim: Differentiate

\[ \Delta v_0 + \eta_0 f(v_0) = 0 \quad \text{and} \quad \Delta \phi_0 + \eta_0 f'(v_0) \phi_0 = 0, \]

we have

\[ (ii) \quad \Delta \nabla v_0 + \eta_0 f'(v_0) \nabla v_0 = 0, \]

\[ (iii) \quad \Delta \nabla \phi_0 + \eta_0 f'(v_0) \nabla \phi_0 + \eta_0 f''(v_0) \phi_0 \nabla v_0 = 0. \]

Inner-product \((ii)\) with \(\nabla \phi_0\), \((iii)\) with \(\nabla v_0\), subtract and integrate on \(\Omega\), one can get

\[
\eta_0 \int_{\Omega} f''(v_0)|\nabla v_0|^2 \phi_0 \, dx
\]
\[ = \int_{\Omega} \left[ \nabla (\Delta v_0) \cdot \nabla \phi_0 - \nabla (\Delta \phi_0) \cdot \nabla v_0 \right] \, dx
\]
\[ = \int_{\partial \Omega} \left[ \Delta v_0 (\nabla \phi_0 \cdot \nu) - \Delta \phi_0 (\nabla v_0 \cdot \nu) \right] \, ds
\]
\[ = - \eta_0 \int_{\partial \Omega} f(v_0) (\nabla \phi_0 \cdot \nu) \, ds + \eta_0 \int_{\partial \Omega} f'(v_0) \phi_0 (\nabla v_0 \cdot \nu) \, ds
\]
\[ = - \eta_0 f(0) \int_{\partial \Omega} \nabla \phi_0 \cdot \nu \, ds. \]

Since \(f(0) = 0\), we have

\[
\int_{\Omega} f''(v_0)|\nabla v_0|^2 \phi_0 \, dx = 0
\]
i.e.

\[
\int_0^1 r^{n-1} f''(v_0(r)) [v_{0r}(r)]^2 \phi_0(r) \, dr = 0.
\]

Since \(\phi_0(r) > 0\), \([v_{0r}(r)]^2 > 0\), \(r \in (0, 1]\); \(f''(v_0(r))\) must change sign.

Note that \(f''(u) = u^{p-2}[p(p - 1) - (p + 1)pu]\); \(f''(u) = 0\) for \(u = \frac{p-1}{p+1}\); \(f''(u) > 0\) for \(0 < u < \frac{p-1}{p+1}\). Then there exists \(r_0 \in (0, 1)\) such that

\[
f''(v_0(r)) \leq 0, \quad \text{in} \ [0, r_0]
\]
\[
f''(v_0(r)) \geq 0, \quad \text{in} \ [r_0, 1].
\]

Next we claim: There exists \(k > 0\) such that

\[
k \phi_0(r) \geq -v_{0r}(r), \quad r \in [0, r_0]
\]
\[
k \phi_0(r) \leq -v_{0r}(r), \quad r \in [r_0, 1]
\]

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Indeed, let \( v = \phi_0 + v_{0r} \), then \( v(0) = \phi_0(0) > 0 \), \( v(1) = v_{0r}(1) < 0 \). So there is at least one point \( t_0 \in (0, 1) \) such that \( v(t_0) = 0 \). We will prove that \( t_0 \) is unique. Since \( v_{0r} \) satisfies
\[
v''_{0r} + \frac{n-1}{r}v'_{0r} - \frac{n-1}{r^2}v_{0r} + \lambda f'(v_0)v_{0r} = 0
\]
and \( \phi_0 \) satisfies
\[
(iv) \quad \phi''_0 + \frac{n-1}{r}\phi'_0 + \lambda f'(v_0)\phi_0 = 0,
\]
so we have
\[
(v) \quad v'' + \frac{n-1}{r}v' + \lambda f'(v_0)v = \frac{n-1}{r^2}v_{0r}.
\]
Multiply \((iv)\) by \( r^{n-1}v \), \((v)\) by \( r^{n-1}\phi_0 \), and subtract, we obtain
\[
(vi) \quad \left[ r^{n-1}(\phi'_0 v - v'_0) \right]' = -(n-1)r^{n-3}\phi_0 v_{0r}.
\]
Suppose \( v(r) \) has more than one zeros in \((0, 1)\), and denote \( t_1 < t_2 \) the last two zeros of \( v(r) \). Then \( v'(t_1) \geq 0 \), \( v'(t_2) \leq 0 \). Integrate \((vi)\) over \((t_1, t_2)\) we have
\[
-t_2^{n-1}v'(t_2)\phi_0(t_2) + t_1^{n-1}v'(t_1)\phi_0(t_1)
\]
\[
= - (n-1) \int_{t_1}^{t_2} r^{n-3}\phi_0(t) v_{0r}(t) dt,
\]
where the left-hand side \( \leq 0 \), but the right-hand side \( > 0 \). This is a contradiction, and that is to say \( v(r) = 0 \) has only one root in \((0, 1)\). In other words, there is only one \( t_0 \in (0, 1) \) such that
\[
\phi_0(t_0) = -v_{0r}(t_0).
\]
Since \( \phi_0(r) \) satisfies a linear differential equation, by varying the coefficient \( k \), we can finish the proof of the claim above.

From this claim, we have
\[
k^2 \int_0^1 r^{n-1}f''(v_0(r))\phi_0^2(r)dr
\]
\[
= \int_0^1 r^{n-1}f''(v_0(r))[k\phi_0(r)]^2\phi_0(r)dr
\]
\[
< \int_0^1 r^{n-1}f''(v_0(r))v_{0r}^2(r)\phi_0(r)dr
\]
\[
= 0.
\]
recall Lemma 3.1 and the representation (i), then \( \tau''(0) > 0 \). ♦

Through these preparations we obtain the second main result:

**Theorem 3.2** Suppose \( k = 0 \). If \( n = 1, 2, 1 < p < \infty \); or \( n \geq 3, 1 < p < \frac{4}{n+2} \), then there exists \( \eta_0 > 0 \) such that (3.4) has exactly two positive solutions if \( \eta > \eta_0 \), exactly one positive solution if \( \eta = \eta_0 \), and no positive solution if \( \eta < \eta_0 \), i.e. the bifurcation diagram of (3.4) is exactly ”-shaped.

Furthermore, all positive solutions of (3.4) lie on a single smooth solution curve in the space \( R_{+} \times C^2(\Omega) \), which consists of two branches \( v_*(x, \eta) < v^*(x, \eta) \) for \( \eta > \eta_0 \). The mapping \( \eta \mapsto v^*(x, \eta) \) is continuous and increasing if \( |x| < 1 \), \( \lim_{\eta \to \infty} v^*(x, \eta) = 1 \); the mapping \( \eta \mapsto v_*(x, \eta) \) is continuous and decreasing if \( |x| < 1 \), \( \lim_{\eta \to \infty} v_*(x, \eta) = 0 \); \( \lim_{\eta \to \eta_0+0} v_*(x, \eta) = \lim_{\eta \to \eta_0+0} v_*(x, \eta) = v(x, \eta_0) \) if \( |x| < 1 \).

**Proof.** This result is mentioned in the Theorem 1.3 in [OS1], but is not proved. As reference in our proof for main result, we give a direct proof. Here \( \lim_{\eta \to \infty} v_*(0, \eta) = 0 \) is a new result.

By Lemma 3.2, there exists an \( \eta_1 > 0 \) such that (3.4) has at least one positive solution if \( \eta > \eta_1 \). Set \( \eta_0 = \inf \{ \eta > 0 : (3.4) \text{ has at least a positive solution} \} \), we can claim that \( \eta_0 > 0 \). Since \( f(u)/u = u^{p-1}(1-u) \) is continuous for \( u \in [0, 1] \), so there exists a \( k > 0 \) such that \( f(u) \leq ku \) for \( u(r) \in [0, 1] \). If \( v(x) \) is a positive solution of (3.4) for \( \eta > 0 \), i.e.

\[
\begin{align*}
\Delta v + \eta f(v) &= 0, \quad \text{in } \Omega \quad (A) \\
v(x) &= 0, \quad \text{on } \partial \Omega \quad (B).
\end{align*}
\]

Recall Lemma 2.2, we have \( 0 \leq v(x) \leq 1 \) for any \( x \in \Omega \). Multiply (A) by \( v \), and integrate over \( \Omega \), we get \( \int_{\Omega} v_\Delta v - \eta f(v) v \, dx = 0 \). Integrate by parts, we have

\[
\int_{\Omega} |\nabla v(x)|^2 \, dx = \eta \int_{\Omega} f(v(x)) v(x) \, dx 
\leq \eta k \int_{\Omega} v^2(x) \, dx \quad (C)
\]

Let \( \lambda_1 \) be the principal eigenvalue of \(-\Delta\). By the Variational Principle for the
principal eigenvalue (see [LE], page 336, Theorem 2), we have $\lambda_1 > 0$ and
\[
\int_{\Omega} |\nabla v|^2 dx \geq \lambda_1 \int_{\Omega} v^2 dx. \quad (D)
\]
Since $v \in C^{2,\alpha}_0(\Omega) \subset H^1_0(\Omega)$, combine (C) and (D), we obtain
\[
\lambda_1 \int_{\Omega} v^2 dx \leq \eta k \int_{\Omega} v^2 dx.
\]
Since $v(x)$ is a positive solution of (3.4), we have
\[
\eta \geq \frac{\lambda_1}{k}.
\]
Hence $\eta_0 \geq \frac{\lambda_1}{k} > 0$.

Let $\eta_n > \eta_0$, $\eta_n \to \eta_0$ as $n \to \infty$, and $v_n$ be the corresponding positive solution of (3.4), i.e.
\[
\begin{aligned}
\Delta v_n + \eta_n f(v_n) &= 0, \quad \text{in } \Omega, \\
v_n &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]
By the Boundary $H^2$-regularity of Elliptic Equation (see [LE], page 317, Theorem 4), we know that $v_n \in H^2(\Omega)$, and
\[
\|v_n\|_{H^2(\Omega)} \leq C \left( \|\eta_n f(v_n)\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Omega)} \right).
\]
Since $0 \leq v_n(x) \leq 1$, there must exist an $M$ such that
\[
\|v_n\|_{H^2(\Omega)} \leq M.
\]
By the Soblev Embedding Theorem, we may assume that $v_n \to v_0$ in $H^1(\Omega)$ as $n \to \infty$. For any $\phi \in H^1(\Omega)$, we have
\[
\int_{\Omega} \nabla v_n \cdot \nabla \phi dx = \eta_n \int_{\Omega} f(v_n) \phi dx.
\]
Let $n \to \infty$, we obtain
\[
\int_{\Omega} \nabla v_0 \cdot \nabla \phi dx = \eta_0 \int_{\Omega} f(v_0) \phi dx,
\]
i.e. $v_0 \in H^1(\Omega)$ is a weak solution of the problem
\[
\begin{aligned}
\Delta v + \eta_0 f(v) &= 0, \quad \text{in } \Omega, \\
v &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]
Again by the regularity, we know \( v_0 \in C^2(\Omega) \). In the following, we may see \( v_0(0) > 0 \), so by the Strong Maximum Principle, \( v_0(x) > 0 \), \( x \in \Omega \). i.e. (3.4) has at least a positive solution, we may choose one of them and again denote by \( v_0(x) \). If (3.4) has another positive solution \( v \) with \( \eta = \eta_0 \), and \( v(0) = v_0(0) \), we have \( v(x) = v_0(x) \) from Lemma 3.2. By the definition of \( \eta_0 \), (3.4) has no solution if \( \eta < \eta_0 \).

At \( \eta = \eta_0 \), the positive solution \( v_0(x) \) of (3.4) must be a degenerate solution. If not, then by Implicit Function Theorem, there would be a positive solution of (3.4) for \( \eta < \eta_0 \), and this contradicts the definition of \( \eta_0 \). By Proposition 3.3 in [OS1], the solution \( \phi \) of linearized problem (3.6) does not change sign in \( \Omega \). By Lemma 3.3, all solutions of (3.4) near \( (\eta_0, v_0) \) have the form \((\eta_0 + \tau(s), v_0 + s\phi + z(s))\), with \( \tau(0) = \tau'(0) = 0 \), \( z(0) = z'(0) = 0 \), and \( \tau''(0) > 0 \). So the solution curve "turn right" at \((\eta_0, v_0)\). We may denote the upper and lower branches by \( v^*(\cdot, \eta) \) and \( v_*'(\cdot, \eta) \) respectively for \( \eta > \eta_0 \). That is to say

\[
\n v^*(x, \eta) > v_0(x) > v_*(x, \eta), \quad |x| < 1,
\]

hence \( v_0(0) > 0 \) since \( v_*(\cdot, \eta) \) is a positive solution of (3.4). Denote \( v^*(r, \eta) = v^*(x, \eta), v_*(r, \eta) = v_*(x, \eta) \) for \( r = |x| \leq 1 \).

As long as \((\eta, v^*(\cdot, \eta))\) and \((\eta, v_*(\cdot, \eta))\) are non-degenerate for \( \eta > \eta_0 \), the Implicit Function Theorem will ensure that we can extend continuously the two branches in the direction of increasing \( \eta \), and we still denote the extension as \( v^*(\cdot, \eta) \) and \( v_*(\cdot, \eta) \), since \( \|v^*(\cdot, \eta)\|_\infty \leq 1, \|v_*(\cdot, \eta)\|_\infty \leq 1 \) for \( \eta > \eta_0 \). The process of continuation towards larger value of \( \eta \) for both branches may be stopped at some finite \( \eta^* \) by one of the followings:

\[
\begin{cases} 
(i) \|v^*(\cdot, \eta_n)\|_\infty \text{ or } \|v_*(\cdot, \eta_n)\|_\infty \text{ goes to } 0 \text{ for some } \eta_n \to \eta^* - 0; \\
(ii) v^*(\cdot, \eta^*) \text{ or } v_*(\cdot, \eta^*) \text{ is a degenerate solution.}
\end{cases}
\]

(i) cannot occur. Otherwise, denoting \( v_n = v^*(\cdot, \eta_n) \) or \( v_*(\cdot, \eta_n) \). Let \( \lambda_1 > 0 \) be the principal eigenvalue of \(-\Delta\), and \( \phi_1 > 0 \) (otherwise we take \(-\phi_1\)) be the corresponding principal eigenfunction. Multiplying \( 0 = -\Delta v_n - \eta_n (1 - v_n) v_n^p \) by \( \phi_1 \) and integrating by parts, we have a contradiction

\[
0 = \int_\Omega \left[ \lambda_1 \phi_1 - \eta_n (1 - v_n) v_n^{p-1} \right] v_n dx \to \int_\Omega \lambda_1 \phi_1 dx > 0, \quad \text{as } n \to \infty.
\]
Finally, (ii) can not occur. Suppose \((\eta^*, v^*(\cdot, \eta^*))\) is a degenerate solution, then by Lemma 3.3, all solutions near \((\eta^*, v^*(\cdot, \eta^*))\) must lie to its right side, which is a contradiction. Therefore we can always extend these two branches of solution curve to \(\eta = \infty\).

We denote by \(v^*_\eta (r, \eta)\) the derivative of \(v^*(r, \eta)\) with respect to \(\eta\), then we show \(v^*_\eta (r, \eta) > 0\) for any \(\eta > \eta_0\) and any \(r \in (0, 1)\). Let

\[
\eta_1 = \sup \{ \eta > \eta_0 : v^*_\eta (r, \eta) > 0, \text{ for any } r \in (0, 1) \}.
\]

If \(\eta_1 < \infty\), then there are two possible cases:

Case (i) \(v^*_\eta (r, \eta_1) \geq 0\) for all \(r \in (0, 1)\) and \(v^*_\eta (r_1, \eta_1) = 0\) for some \(r_1 \in (0, 1)\). Then \(r_1\) is a minimum point of \(v^*_\eta (r, \eta_1)\), so \(v^*_\eta (r_1, \eta_1) = 0\). Now we will drop the upper-script “*” from \(v^*(r, \eta_1)\) for convenience. \(v_\eta\) satisfies

\[
\begin{cases}
  v''_\eta + \frac{n-1}{r} v'_\eta + \eta f'(v) v_\eta + f(u) = 0, \ r \in (0, 1) \\
  v'_\eta(0) = v_\eta(1) = 0.
\end{cases}
\]  

(A)

On the other hand, \(v_r\) satisfies

\[
\begin{cases}
  v''_r + \frac{n-1}{r} v'_r + \eta f'(v) v_r - \frac{n-1}{r^2} v_r = 0, \ r \in (0, 1) \\
  v_r(0) = 0.
\end{cases}
\]  

(B)

Multiply (A) by \(r^{n-1} v_\eta\), (B) by \(r^{n-1} v_r\), subtract and integrate from 0 to \(r_1\), we obtain

\[
\left. r^{n-1} [v'_\eta v_r - v_\eta v'_r] \right|_0^{r_1} = \int_0^{r_1} \left[ - (n-1) r^{n-3} v_\eta v_r - r^{n-1} f'(v) v_r \right] dr, \quad (C)
\]

The left-hand side is 0, but the right-hand side is positive. So this is a contradiction.

Case (ii) \(v^*_\eta (1, \eta_1) = 0\). Use (C) with \(r_1 = 1\), we will get a same contradiction.

This means that \(\eta_1 = \infty\). So \(v^*_\eta (r, \eta) > 0\) for \(\eta > \eta_0\) and \(r \in (0, 1)\).

Similar to upper branch, we can show that the lower branch solution curve is decreasing with respect to \(\eta\) (see [KLO1]). Hence

\[
v^*(r, \eta) > v_0(r) > v_\eta (r, \eta)
\]
for any $r \in [0, 1]$ and $\eta \in (\eta_0, \infty)$, and
\[
\begin{cases}
\lim_{\eta \to \infty} v^*(r, \eta) = \xi(r) \in [v_0(r), 1] \\
\lim_{\eta \to \infty} v_*(r, \eta) = \eta(r) \in [0, v_0(r)]
\end{cases}
\]
for all $r \in [0, 1)$. By the same proof as in the proof of Theorem 1.1 in [OS1], we see that
\[
\xi(r) \equiv 1, \quad |r| < 1, \quad \xi(1) = 0,
\]
and
\[
\eta(r) \equiv 0, \quad 0 < |r| < 1.
\]
Now we prove that $\eta(0) = 0$. Suppose $\alpha = \eta(0) > 0$, i.e.
\[
\lim_{\eta \to \infty} v_*(0, \eta) = \alpha
\]
then $\alpha < v_0(0) < 1$. Consider the initial value problem of differential equation:
\[
\begin{cases}
(r^{n-1}z'(r))' = -r^{n-1}(1 - z(r))z^p(r), \quad r > 0 \\
z(0) = \alpha \\
z'(0) = 0.
\end{cases}
\]
(E)
If $z(r) > 0$ for $r \in (0, r_0)$, then by integrating (E) from 0 to $r$, we have
\[
\begin{align*}
 r^{n-1}z'(r) &= - \int_0^r s^{n-1}(1 - z(s))z^p(s)ds \\
&< 0,
\end{align*}
\]
so $z'(r) < 0$ for $r \in (0, r_0)$. But we say $r_0$ is a finite number. If not, i.e. $r_0 = \infty$, then $z(x) = z(|x|) = z(r)$ satisfies
\[
\Delta z(x) = -(1 - z(x))z^p(x) \leq 0, \quad on \ R^n,
\]
since $0 \leq z(x) \leq 1$. Hence $z(x)$ is a bounded sub-harmonic function on $R^n$, therefore $z(x) \equiv constant$ for any $x \in R^n$. Clearly this is impossible. Thus $0 < r_0 < \infty$, $z(r) > 0$ for $r$ in $[0, r_0)$, and $z(r_0) = 0$.
By the continuous dependence of the solution on the initial values, for \( \eta^* \)
large enough, e.g. \( \eta^* > 3r_0 \), the unique solution \( z^* \) of the initial value problem

\[
\begin{cases}
(r^{n-1}z'(r))' = -r^{n-1}(1 - z(r))z^p(r), \ r > 0 \\
z(0) = v_*(0, \eta_*) \\
z'(0) = 0
\end{cases}
\]

has a first zero point \( r^* < 2r_0 \) close to \( r_0 \).

Set \( v(r) = z^*(r^*r) \), then \( v'(0) = 0 \), \( v(1) = z^*(r^*) = 0 \), and

\[
[r^{n-1}v'(r)]' = [r^*r^{n-1}z'(r^*r)]'
= -(r^*r^{n-1})[1 - z(r^*r)]z^p(r^*r)
= -\eta r^{n-1}[1 - v(r)]v^p(r), \ r \in (0, 1)
\]

i.e. \( v(r) \) satisfies (3.4) with \( \eta = r^* \) and \( v(0) = v_*(0, \eta_*) \). By Lemma 3.3, we have
\( v(r) = v_*(\cdot, \eta_*) \) and \( \eta = \eta_* \). But it contradicts with \( \eta = r^* < 2r_0 < 3r_0 < \eta_* \).
Thus \( \alpha = 0 \), and we finish the proof. \( \diamond \)
Chapter 4  Perturbed Equation and The S-shaped Bifurcation Curve

Now we can illustrate our main result:

**Theorem 4.1** Suppose $n = 1$ or $2$. Then there exists $k_0 > 0$ such that for any $k \in (0, k_0)$ the bifurcation diagram of (3.3) is exactly "S"-shaped. More precisely, there exists $0 < \lambda_* < \lambda^* < \infty$ such that (3.3) has exactly three positive solutions if $\lambda_* < \lambda < \lambda^*$; has exactly one positive solution if $\lambda > \lambda^*$ or $\lambda < \lambda_*$; and has exactly two positive solutions if $\lambda = \lambda^*$ or $\lambda = \lambda_*$. 

**Proof.** By [GNN], the positive solutions $u(x)$ of (3.3), and $v(x)$ of (3.4) are radially symmetric: 

$$
\begin{align*}
&u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u(r) + k) = 0, \quad r \in (0, 1) \\
&u'(0) = u(1) = 0
\end{align*}
$$

(1)

and $v(x) = v(r), \quad r = |x|$

$$
\begin{align*}
&v''(r) + \frac{n-1}{r}v'(r) + \eta f(v(r)) = 0, \quad r \in (0, 1) \\
&v'(0) = v(1) = 0
\end{align*}
$$

(2)

For $\eta > \eta_0$, we denote by $v_*(r, \eta) < v^*(r, \eta)$ the lower and upper branches of solution curve of (2). Define 

$$
v_0(r) = v^*(r, \eta_0) = v_*(r, \eta_0) = v_0(r, \eta_0) \quad \text{for} \quad r \in [0, 1].
$$

For any $k \in (0, v_0(0))$ and $\eta \geq \eta_0$, since $\partial v^*(r, \eta)/\partial r < 0$ for arbitrary $r \in (0, 1]$, there must exist an unique $a^* = a^*(k, \eta) \in (0, 1)$ such that 

$$
v^*(a^*, \eta) = k. \quad (\star)
$$

By the Implicit Function Theorem, $a^*(k, \eta)$ is smooth in both variables since $\partial v^*/\partial r < 0$ for $r \in [0, 1]$. 

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It is easily checked that
\[ u = u^*(r, \eta, k) \equiv v^*(a^*(k, \eta)r, \eta) - k \]
is a positive solution of (1) with
\[ \lambda = \lambda^*(k, \eta) \equiv [a^*(k, \eta)]^2\eta. \]
For each \( k \in (0, v_0(0)) \), we can conclude that \( v_*(0, \eta) \) is continuous and decreasing with respect to \( \eta \) from Theorem 3.2. That is to say
\[ v_0(0) = v_*(0, \eta_0) > v_*(0, \eta) > 0 \quad \text{for} \quad \eta > \eta_0, \]
so there is an unique \( \eta_k > \eta_0 \) such that
\[ v_*(0, \eta_k) = k. \]
For each \( \eta \in (\eta_0, \eta_k) \), \( \partial v_*(r, \eta)/\partial r < 0 \) for \( r \in (0, 1) \) and
\[ v_*(1, \eta) = 0 < k = v_*(0, \eta_k) < v_*(0, \eta), \]
so we can find an unique \( a_* = a(k, \eta) \in (0, 1) \) such that \( v_*(a_*, \eta) = k. \) Thus
\[ u = u_*(r, \eta, k) \equiv v_*(a_*(k, \eta)r, \eta) - k \]
is another positive solution of (1) with
\[ \lambda = \lambda_*(k, \eta) \equiv [a_*(k, \eta)]^2\eta. \]
Note that \( v_*(r, \eta_k, k) = 0 \) and \( a_*(k, \eta_k) = 0 \) since \( v_*(0, \eta_k) = k. \) Hence
\[ \lambda_*(k, \eta_k) = 0. \]
Now we define
\[ \Gamma_* = \left\{ (\lambda_*(k, \eta), u_*(\cdot, \eta, k)) : \eta \in [\eta_0, \eta_k] \right\}, \]
then \( \Gamma_* \) is a piece of solution curve of (1), which joints \((0, 0)\) and \((\lambda_k, u_k)\), where
\[
\begin{cases}
\lambda_k = \lambda^*(k, \eta_0) = \lambda_*(k, \eta_0) \\
u_k(r) = u_*(r, \eta_0, k) = u^*(r, \eta_0, k).
\end{cases}
\]
Similarly,
\[ \Gamma^* = \left\{ (\lambda^*(k, \eta), u^*(\cdot, \eta, k)) : \eta \in [\eta_0, \infty) \right\} \]
is a piece of solution curve of (1), which joints \((\lambda_k, u_k)\) to \((\infty, 1 - k)\). Thus \(\Gamma = \Gamma^* \cup \Gamma_*\) is a continuous solution curve of (1) that joints \((0, 0)\) to \((\infty, 1 - k)\). By Lemma 3.3, \(\Gamma\) contains all the solutions of (1).

At first, we analyze the curve \(\Gamma\). Recall (\(*\)) the definition of \(a^*\):

\[ v^*(a^*(k, \eta), \eta) = k. \]

Since \(\partial v^*/\partial r < 0\) for \(r \in (0, 1]\), and \(\partial v^*/\partial \eta > 0\) for \(\eta > \eta_0\) and \(0 \leq r < 1\) we have

\[ \frac{\partial a^*}{\partial \eta} = -\frac{\partial v^*/\partial \eta}{\partial v^*/\partial r} > 0 \]

for \(r \in [0, 1]\) and \(\eta > 0\). Therefore, \(\eta \mapsto \lambda^*(k; \eta)\) is strictly increasing. This implies that \(\Gamma^*\) is a smooth curve which can be parameterized by \(\lambda\).

Next we analyze \(\Gamma_*\) and assume that \(k > 0\) is small.

By Theorem 3.2, \(v_*(0, \eta)\) is decreasing to 0 as \(\eta \to \infty\). Hence we may take \(\eta_1 > \eta_0\) such that \(v_*(0, \eta) < \frac{1}{2} \left( \frac{p-1}{p+1} \right)\) for \(\eta > \eta_1\). For convenience, we drop the subscript * from \(v_*, a_*, \lambda_*\) in the following.

From \(v(a(k, \eta), \eta) = k\), we can work out

\[ \frac{\partial a(k, \eta)}{\partial \eta} = -\frac{\partial v(a(k, \eta), \eta)/\partial \eta}{\partial v(a(k, \eta), \eta)/\partial r}, \]

and hence

\[ \frac{\partial \lambda(k, \eta)}{\partial \eta} = a(k, \eta) \left[ a(k, \eta) - 2\eta \frac{\partial v(a(k, \eta), \eta)/\partial \eta}{\partial v(a(k, \eta), \eta)/\partial r} \right]. \]

Since \(v(0, \eta)\) decreases to 0 as \(\eta \to \infty\), and \(v(0, \eta_k) = k\), we know that

(a) \(\eta_k \to \infty\) as \(k \to 0\).

Note \(v(1, \eta) = 0\) and \(v(a(k, \eta), \eta) = k\) for all \(\eta \in [\eta_0, \eta_1]\) and \(\partial a(k, \eta)/\partial k = -[\partial v/\partial \eta]/[\partial v/\partial r] > 0\) for \(\eta \in [\eta_0, \eta_1]\), we see that

(b) \(a(k, \eta) \to 1\) as \(k \to 0\) uniformly for \(\eta \in [\eta_0, \eta_1]\).

Also can we obtain

(c) \(\frac{\partial v(a(k, \eta), \eta)}{\partial \eta} \to \frac{\partial v(1, \eta)}{\partial \eta} \equiv 0\),

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as \( k \to 0 \) uniformly for \( \eta \in [\eta_0 + \delta, \eta_1] \), \( \delta \in (0, \eta_1 - \eta_0) \).

\[
(d) \quad \frac{\partial(a(k, \eta), \eta)}{\partial r} \longrightarrow \frac{\partial v(1, \eta)}{\partial r} \leq C_0 < 0,
\]
as \( k \to 0 \) uniformly for \( \eta \in [\eta_0, \eta_1] \). We deduce that

\[
\frac{\partial \lambda(k, \eta)}{\partial \eta} \longrightarrow 1
\]
as \( k \to 0 \) uniformly for \( \eta \in [\eta_0, \eta_1] \). There exists \( k_1 > 0 \) such that

\[
\frac{\partial \lambda(k, \eta)}{\partial \eta} > 0
\]
for all \( k \in (0, k_1) \) and \( \eta \in [\eta_0 + \delta, \eta_1] \). This implies that

\[
\Gamma_*^\delta = \left\{ (\lambda_*(k, \eta), u_*(\cdot, \eta, k)) : \eta \in [\eta_0 + \delta, \eta_1] \right\}
\]
can be parameterized by \( \lambda \).

Denote

\[
\tilde{\Gamma}_*^\delta = \left\{ (\lambda_*(k, \eta), u_*(\cdot, \eta, k)) : \eta \in [\eta_0, \eta_0 + \delta] \right\},
\]
we are going to show that if \( k \) is sufficiently small, then \( \tilde{\Gamma}_*^\delta \) contains exactly one degenerate solution of (1), at which the curve "turns to the right". If we finish it,

\[
\Gamma_0 = \Gamma^* \cup \tilde{\Gamma}_*^\delta \cup \Gamma_*^\delta
\]
is a piece of solution curve of (1), which is exactly "C"-shaped with only one turning point on \( \tilde{\Gamma}_*^\delta \).

We will show that if \( k \) is sufficiently small, then near any degenerate solution \((\lambda, u) \in \tilde{\Gamma}_*^\delta \) \((\tilde{\Gamma}_*^\delta \) contains at least one degenerate solution of (1), because the curve \( \Gamma \) has to make a change of direction there), the solutions form a smooth curve which "turns to the right" at \((\lambda, u) \). This implies that there can be only one degenerate solution on \( \tilde{\Gamma}_*^\delta \), and the curve "turns to the right" at that point.

To stress the \( k \) dependence, we denote by \((\lambda^k, u^k)\) an arbitrary degenerate solution of (1) lying on \( \tilde{\Gamma}_*^\delta \). We first show \((\lambda^k, u^k) \to (\eta_0, v_0) \) in \( R \times C^2(\bar{\Omega}) \) as \( k \to 0 \).

Indeed, for any sequence \( k_n \to 0 \), since \( \lambda^{k_n} \) is bounded from above and below by positive constants, we may assume that \( \lambda^{k_n} \to \lambda^0 > 0 \). Using the relationship
between \(u\) and \(v\), we may check easily that \(u^k \to u^0\) in the \(C^2\)-norm, where \(u^0(r) = v_*(r, \lambda^0)\) is a positive solution of (2) with \(\eta = \lambda_0\).

By Theorem 3.1, the linearized eigenvalue problem (3.5) of (1) at \((\lambda^k, u^k)\) has an eigenfunction \(w_k > 0\). Hence it is the principal eigenfunction and is unique if we assume \(\|w_k\|_\infty = 1\).

Using standard Elliptic Regularity and Compact Embedding Theorems, we may assume that

\[ w_k \to w_0 \quad \text{in } C^2 \text{-norm as } n \to \infty. \]

We can easily see that \((\lambda_0, u_0)\) is a degenerate solution of (2) (see the proof in Theorem 3.2), whose principal eigenfunction is \(w_0\) for the corresponding linearized problem. From Theorem 3.2 we know that \((\eta_0, v_0)\) is the only degenerate solution of (2), then \((\lambda^0, u^0) = (\eta_0, v_0)\), i.e.

\[ (\lambda^k, u^k) \to (\eta_0, v_0) \quad \text{in } R \times C^2(\bar{\Omega}) \text{ as } k \to 0, \]

and also \(w_k \to \phi_0\), where \(\phi_0\) is a nontrivial solution of (3.6) with \(\eta = \eta_0\). Next we will use a bifurcation result of Crandall and Rabinowitz Theorem.

Set \(X = C^2_0(\bar{\Omega})\), \(Y = C^0(\bar{\Omega})\) and \(F(\lambda, u) = \Delta u + \lambda f(u + k)\), where \(f(v) = v^p - v^{p+1}\) \((p > 1)\). Then

\[ \langle F_u(\lambda, u), w \rangle = \Delta w + \lambda f'(u + k)w. \]

By Lemma 3.3 we know

\[ N(F_u(\lambda^k, u^k)) = \text{span}\{w_k\} \]

and

\[ \text{codim}R(F_u(\lambda^k, u^k)) = 1 \]

by the Fredholm Alternative. Also

\[ F_\lambda(\lambda^k, u^k) = f(u^k + k) \in R(F_u(\lambda^k, u^k)), \]

since \(\int_{\bar{\Omega}} f(u^k + k)w_k dx > 0\) by Lemma 3.1. Citing the Theorem 3.2 in [CR], one can see the solutions of (1) near the degenerate solution \((\lambda^k, u^k)\) form a smooth curve :

\[ (\lambda_k(s), u_k(s)) = (\lambda^k + \tau_k(s), u^k + sw_k + z_k(s)), \quad (F') \]
where \( s \mapsto (\tau_k(s), z_k(s)) \in R \times Z_k \) is a smooth function near \( s = 0 \) with \( \tau_k(0) = \tau'_k(0) = 0, z_k(0) = z'_k(0) = 0 \), and \( Z_k \) is a complement of \( \text{span}\{w_k\} \) in \( X \).

Substitute the express \((F)\) for the solution \((1)\), differentiate the equation with respect to \( s \) twice at \( s = 0 \), multiply the resulting identity with \( w_k \) and integrate it over \( \Omega \), to obtain
\[
\tau''_k(0) = -\lambda^k \int_\Omega f''(u^k + k)w_k^3 dx \frac{\int_\Omega f(u^k + k)w_k dx}{\int_\Omega f(u^k + k)w_k dx}.
\]
Let \( k \to 0 \), the right-hand side of the above identity converges to
\[
-\eta_0 \int_\Omega f''(v_0)\phi_0^3 dx \int_\Omega f(v_0)\phi_0 dx.
\]
By Lemma 3.3, this quantity is positive. Therefore, for small \( k > 0 \), \( \tau''_k(0) > 0 \) holds. It implies that the solution curve has a ”turn to the right” at \((\lambda^k, u^k)\), provided that \( k > 0 \) is small. This finishes our analysis on \( \Gamma_0 \).

Denote the remaining part of the solution curve \( \Gamma \setminus \Gamma_0 \) by \( \Gamma_1 \). Recalling the choice of \( \eta_1 \), we know \((\lambda^k, u^k) \in \Gamma_1 \) satisfies \( u^k(0) < \frac{1}{2} \left( \frac{p-1}{p+1} \right) \). Now taking \( k_2 = \frac{1}{2} \left( \frac{p-1}{p+1} \right) > 0 \) such that \( k < k_2 \), one can easily get
\[
u^k(0) + k < \frac{p-1}{p+1},
\]
and therefore
\[
f''(u^k(r) + k) > 0, \quad r \in [0, 1].
\]
If \((\lambda^k, u^k) \in \Gamma_1 \) is a degenerate solution of \((1)\), then all the solutions near \((\lambda^k, u^k)\) form a smooth curve
\[
(\lambda_k(s), u_k(s)) = (\lambda^k + \tau_k(s), u^k + sw_k + z_k(s))
\]
with \( \tau_k(0) = \tau'_k(0) = 0, z_k(0) = z'_k(0) = 0 \), where \( s \mapsto (\tau_k(s), z_k(s)) \in R \times Z_k, \ X = \text{span}\{w_k\} \oplus Z_k, \) and \( w_k > 0 \) is the solution of the corresponding linearized problem. We may still obtain \((**\))

Choose \( k_0 = \min\{k_1, k_2\} \), if \( k \in (0, k_0) \), then \( \tau''_k(0) < 0 \). This implies that the solution curve make a ”turn to the left” at such a degenerate solution \((\lambda^k, u^k)\). Since \( \Gamma_1 \) connects the point \((0, 0)\) to \( \Gamma_0 \) in the phase-space, there must exist only one degenerate solution on \( \Gamma_1 \). If there is another one denoted by \((\tilde{\lambda}, \tilde{u})\), with a similar argument as above we will show that all the solutions near \((\tilde{\lambda}, \tilde{u})\) must
lie on the left, which is obviously impossible. That is to say, the curve $\Gamma_1$ is "$\supset$"-shaped.

So, the curve $\Gamma$ in phase-space is "$S$"-shaped, when $k$ is sufficiently small. Denoting the two degenerate solutions on $\Gamma_0$ and $\Gamma_1$ by $(\lambda_*, u_*)$ and $(\lambda^*, u^*)$ respectively, we complete the proof.

\[ \diamond \]

**Acknowledgements.** I wish to thank my tutors Professor Yuwen Wang and associate Professor Junping Shi for positive and patient instructions. In the three postgraduate years, I was endowed with the doctrine of human and nurtured with affluent knowledge by my two tutors. And I am very grateful to their supports and love. So, I can’t render thanks to them without hard works and further achievements. Thanks to my tutors again.

In addition, I want to thank Professor Wen Song, Professor Hui Wang and Professor Shuqin Wang. From their words in hearts during three years, I have learned more than the one in thirty years.
References


1475-1485.


致谢

首先，我要感谢我的导师王玉文教授和史峻平特聘教授。两位老师在我的三年硕士研究生期间，无论是在生活上还是在学习上，都给予了极大的关怀和耐心的教导。特别是在我的毕业论文选题、写作期间，两位老师在百忙之中都要抽出时间，给学生作细致而具体的指导，更令我感激万分。可以说，这几年里学生取得的每一点进步，都倾注着两位老师的一番心血。在今后的求学道路上，我只有通过不断地努力和奋斗，作出更大的成绩，才能报答两位老师对学生的无限恩情。

此外，我还要感谢宋文教授，王辉教授，王书琴教授。三位老师的谆谆教诲，学生将终生牢记。