Uniqueness of positive solutions for a class of semipositone elliptic systems

D.D. Hai*, R. Shivaji

Department of Mathematics, Mississippi State University, Mississippi State, MS 39762, USA

Received 2 March 2005; accepted 9 November 2005

Abstract

We consider the system

\[-\Delta u = \lambda f(v); \quad x \in \Omega\]
\[-\Delta v = \mu g(u); \quad x \in \Omega\]
\[u = 0 = v; \quad x \in \partial \Omega\]

where \(\Omega\) is a ball in \(\mathbb{R}^N\), \(N \geq 1\), and \(\partial \Omega\) is its boundary, \(\lambda, \mu\) are positive parameters, and \(f, g\) are smooth functions that are negative at the origin and \(f(x) \sim x^p\) and \(g(x) \sim x^q\) for large \(x\) for some \(p, q \geq 0\) with \(pq < 1\). We establish the uniqueness of positive solutions when the parameters \(\lambda\) and \(\mu\) are large.

© 2006 Elsevier Ltd. All rights reserved.

1. Introduction

We consider the boundary value problem

\[
\begin{cases}
-\Delta u = \lambda f(v); & x \in \Omega \\
-\Delta v = \mu g(u); & x \in \Omega \\
u = 0 = v; & x \in \partial \Omega
\end{cases}
\]  \hspace{1cm} (1.1)

where \(\Omega\) is the open unit ball in \(\mathbb{R}^N\); \(N \geq 1\), with a smooth boundary \(\partial \Omega\), \(\lambda, \mu\) are positive parameters, and \(f, g\) satisfy the following assumptions:

*Corresponding author. Tel.: +1 601 325 3414; fax: +1 601 325 0005.
E-mail addresses: dang@math.msstate.edu (D.D. Hai), shivaji@math.msstate.edu (R. Shivaji).
(H1) \( f, g : [0, \infty) \to \mathbb{R} \) are nondecreasing \( C^1 \) functions such that \( f(0) < 0 \) and \( g(0) < 0 \) (semipositone system).

(H2) There exist positive numbers \( A, B \) and nonnegative numbers \( p, q \) with \( pq < 1 \) such that
\[
\lim_{x \to \infty} \frac{f(x)}{x^p} = A, \quad \lim_{x \to \infty} \frac{g(x)}{x^q} = B.
\]

(H3) For \( p_1 > p, q_1 > q, \frac{f(x)}{x^{p_1}} \) and \( \frac{g(x)}{x^{q_1}} \) are nonincreasing for \( x \) large.

Our main result is:

**Theorem 1.1.** Let (H1)–(H3) hold. Then there exists a unique positive solution of (1.1) if \( \lambda \) and \( \mu \) are sufficiently large.

In [1] the semipositone \((f(0) < 0)\) single equation
\[
\begin{cases}
-\Delta u = \lambda f(u); & x \in \Omega \\
u = 0; & x \in \partial \Omega
\end{cases}
\]
was studied. In particular for a class of sublinear functions \( f \) at \( \infty \), a uniqueness result was established when \( \lambda \) was sufficiently large. See also [2] where this result was extended to the case when \( \Omega \) is any bounded domain with convex outer boundary. In this paper, we extend the uniqueness result in [1] mentioned above for the system (1.1).

It is well documented in the literature that the study of nonnegative solutions to the semipositone case is mathematically very challenging (see [3]). Recently this was confirmed in [4] in studies of positive solutions to semilinear equations on unbounded domains. By a result in [5] nonnegative solutions of (1.1) are strictly positive in \( \Omega \), and by a result in [6] positive solutions of (1.1) are radially symmetric and decreasing, and thus solve the ordinary differential equations system
\[
\begin{cases}
-(r^{N-1}u')' = \lambda r^{N-1}f(v) \\
-(r^{N-1}v')' = \mu r^{N-1}g(u) \\
u'(0) = 0 = v'(0) \\
u(1) = 0 = v(1).
\end{cases}
\]

Existence of a positive solution when \( \lambda = \mu \) and is sufficiently large was discussed in [7]. The existence part in Theorem 1.1 follows easily by arguments like those used in [7]. Thus in this paper we will concentrate just on the uniqueness part. In particular, we establish the following crucial a priori estimates on positive solutions.

**Theorem 1.2.** Let (H1) and (H2) hold and let \((u, v)\) be a positive solution of (1.3). Then there exist positive constants \( M_i, 1 \leq i \leq 4 \), and \( M \) independent of \( u, v \) such that
\[
M_1(\lambda \mu^p)^{\frac{1}{1-p}} (1-r) \leq u(r) \leq M_2(\lambda \mu^p)^{\frac{1}{1-p}} (1-r), \quad 0 < r < 1
\]
\[
M_3(\mu \lambda^q)^{\frac{1}{1-q}} (1-r) \leq v(r) \leq M_4(\mu \lambda^q)^{\frac{1}{1-q}} (1-r), \quad 0 < r < 1
\]
for \( \lambda, \mu \geq M \).

Such a theorem (a priori estimates) and the corresponding uniqueness result were established for the positone case \((f(0) \geq 0 \text{ and } g(0) \geq 0)\), in [8] when \( \Omega \) is a ball, and in [9] when \( \Omega \) is a general bounded domain. In this paper we will study the more challenging semipositone case \((f(0) < 0 \text{ and } g(0) < 0)\) when \( \Omega \) is a ball. We will concentrate just on extending the
a priori estimates. Once such a priori estimates are established the uniqueness result follows via arguments described in [8]. Thus in this paper we will concentrate just on extending these a priori estimates for the semipositone case $f(0) < 0$ and $g(0) < 0$. Not surprisingly this extension (Theorem 1.2) for the semipositone case is non-trivial and challenging.

We will prove Theorem 1.2 in Section 2.

2. Proof of Theorem 1.2

Here we will consider the case when $p > 0$ and $q > 0$. When $p = 0$ or $q = 0$ the proof follows by similar arguments.

Let $(u, v)$ be a positive solution of (1.3). Let $\beta_1, \theta_1$ be the positive zeros of $g$ and $G$ respectively, and $\beta_2, \theta_2$ be the positive zeros of $f$ and $F$ respectively where $G(s) := \int_0^s g(t)dt$ and $F(s) := \int_0^s f(t)dt$. Let $\rho_1 \in (\beta_1, \theta_1)$, $\rho_2 \in (\beta_2, \theta_2)$, $r_1 \in (0, \beta_1)$ and $r_2 \in (0, \beta_2)$ be fixed.

We will first establish the following claim.

Lemma 2.1. There exists a constant $m > 0$ such that $u\left(\frac{1}{2}\right) \geq \rho_1$ or $v\left(\frac{1}{2}\right) \geq \rho_2$ if $\lambda, \mu \geq m$.

Proof. Assume not. Then $u\left(\frac{1}{2}\right) < \rho_1$ and $v\left(\frac{1}{2}\right) < \rho_2$. Let $c_1 := \sup_{[0,r_1]} g(x)$. Clearly $c_1 < 0$.

If $u\left(\frac{3}{4}\right) \leq r_1$ then

$$-(r^{N-1} v'(r))' = \mu g(u(r)) r^{N-1} \leq \mu c_1 r^{N-1}$$

for $r \in \left[\frac{3}{4}, 1\right]$. \hfill (2.1)

Integrating (2.1) from $r$ to 1 we obtain

$$-v'(1) + r^{N-1} v'(r) \leq \mu c_1 \left[\frac{1}{N} - \frac{r^N}{N}\right]$$

and hence

$$-v'(r) \geq -\frac{\mu c_1}{N r^{N-1}} [1 - r^N] \geq -\frac{\mu c_1}{N} [1 - r]$$

for $r \in \left[\frac{3}{4}, 1\right]$. \hfill (2.2)

Integrating (2.2) from $\frac{3}{4}$ to 1 we obtain

$$r_2 > v\left(\frac{3}{4}\right) \geq -\frac{\mu c_1}{N} \left(\frac{1}{4}\right)^{\frac{2}{2}}. \quad (2.3)$$

Hence if $\mu > \frac{32 N r_2}{c_1}$ then (2.3) is a contradiction. Similarly if $\lambda > \frac{32 N \rho_1}{c_2}$ where $c_2 := \sup_{[0,r_2]} f(x)$ then $v\left(\frac{3}{4}\right) > r_2$. Thus there exists a constant $\tilde{m} > 0$ such that if $\lambda, \mu \geq \tilde{m}$ then

$$r_1 < u\left(\frac{3}{4}\right) \leq u\left(\frac{1}{2}\right) < \rho_1 \quad (2.4)$$
and
\[ r_2 < v\left(\frac{3}{4}\right) \leq v\left(\frac{1}{2}\right) < \rho_2. \]

Let \( H(r) := u'(r)v'(r) + \lambda F(v(r)) + \mu G(u(r)) \). Then \( H(1) = u'(1)v'(1) \geq 0 \) and

\[
\frac{dH}{dr} = u''v' + u'v'' + \lambda f(v)v' + \mu g(u)u' \\
= u'[v'' + \mu g(u)] + v'[u'' + \lambda f(v)] \\
= u'\left[-\frac{(N-1)v''}{r}\right] + v'\left[-\frac{(N-1)\mu'}{r}\right] \\
= -\frac{2(N-1)\mu'v'}{r} \leq 0.
\]

Thus \( H(r) \geq 0 \). Further,

\[
(u')^2 + (v')^2 \geq 2u'v' \\
\geq -2\lambda F(v) - 2\mu G(u) \\
\geq -2\lambda K_2 - 2\mu K_1 \quad \text{for } r \in \left(\frac{1}{2}, \frac{3}{4}\right)
\]

where \( K_1 = \sup_{[r_1, r_2]} G(x) \) and \( K_2 = \sup_{[r_1, r_2]} F(x) \). Note that \( K_1 < 0 \) and \( K_2 < 0 \). Hence

\[
[-u' + (-v')]^2 \geq -2\lambda K_2 - 2\mu K_1 \quad \text{and}
\]

\[
-u' - v' \geq \sqrt{-2\lambda K_2 - 2\mu K_1} \quad \text{for } r \in \left(\frac{1}{2}, \frac{3}{4}\right). \tag{2.6}
\]

Integrating (2.6) from \( \frac{1}{2} \) to \( \frac{3}{4} \) we obtain

\[
-u\left(\frac{3}{4}\right) + u\left(\frac{1}{2}\right) - v\left(\frac{3}{4}\right) + v\left(\frac{1}{2}\right) \geq \sqrt{-2\lambda K_2 - 2\mu K_1}\left(\frac{1}{4}\right). \tag{2.7}
\]

By (2.4) and (2.5), the L.H.S. of (2.7) is bounded if \( \lambda, \mu \geq \bar{m} \) while the R.H.S. of (2.7) \( \to \infty \) as \( \lambda \) or \( \mu \to \infty \). Hence Lemma 2.1 is proven. \( \square \)

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \lambda, \mu \geq m \) and suppose \( u\left(\frac{1}{4}\right) \geq \rho_1 \). Then

\[
v'(r) = -\frac{\mu}{r^{N-1}} \int_0^r s^{N-1} g(u)ds \leq -\frac{\mu\delta r}{N} \quad \text{for } r \leq \frac{1}{2}, \tag{2.8}
\]

where \( \delta := g(\rho_1) > 0 \). Integrating (2.8) from \( \frac{1}{4} \) to \( \frac{1}{2} \) we obtain

\[
v\left(\frac{1}{2}\right) - v\left(\frac{1}{4}\right) \leq -\frac{\mu\delta}{N} \left\{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right\}. \]

Hence

\[
v\left(\frac{1}{4}\right) \geq \epsilon_3 \mu \tag{2.9}
\]
where $c_3 := \frac{\delta}{N} \left\{ \frac{(\delta)^2}{2} - \frac{(\delta)^2}{2} \right\} > 0$. Now

$$u(r) = \lambda \int_r^1 \frac{1}{s^{N-1}} \int_0^s \tau^{N-1} f(\tau) d\tau ds.$$ 

Thus

$$u \left( \frac{1}{4} \right) = \lambda \int_{\frac{1}{4}}^1 \frac{1}{s^{N-1}} \left[ \int_0^{\frac{1}{4}} \tau^{N-1} f(\tau) d\tau + \int_{\frac{1}{4}}^s \tau^{N-1} f(\tau) d\tau \right] ds$$

$$\geq \lambda \int_{\frac{1}{4}}^1 \frac{1}{s^{N-1}} \left[ \left( \int_0^{\frac{1}{4}} \tau^{N-1} d\tau \right) f \left( v \left( \frac{1}{4} \right) \right) + f(0) \right] ds$$

$$= \lambda \left[ \alpha_1 f \left( v \left( \frac{1}{4} \right) \right) + f(0) \right] \alpha_2$$ (2.10)

where $\alpha_1 = \int_{\frac{1}{4}}^1 \tau^{N-1} d\tau > 0$ and $\alpha_2 = \int_{\frac{1}{4}}^1 s^{N-1} ds > 0$. Now $\lim_{s \to \infty} f(s) = \infty$ and $\lim_{s \to \infty} \frac{f(s)}{s^p} = A > 0$. Hence by (2.9) there exists a constant $\overline{m} > 0$ such that if $\mu \geq \overline{m}$ then

$$\alpha_1 f \left( v \left( \frac{1}{4} \right) \right) + f(0) \geq \frac{\alpha_1}{2} f \left( v \left( \frac{1}{4} \right) \right) \geq \frac{\alpha_1}{2} A \left[ v \left( \frac{1}{4} \right) \right]^p.$$ 

Then (2.10) implies

$$u \left( \frac{1}{4} \right) \geq \lambda \frac{\alpha_1 \alpha_2 A}{4} \left[ v \left( \frac{1}{4} \right) \right]^p.$$ (2.11)

In particular, by (2.9) we can assume w.l.o.g. that $v \left( \frac{1}{4} \right) \geq 1$ and hence

$$u \left( \frac{1}{4} \right) \geq c_4 \lambda$$ (2.12)

where $c_4 := \frac{\alpha_1 \alpha_2 A}{4}$. Now using $v \left( \frac{1}{4} \right) = \mu \int_{\frac{1}{4}}^1 \frac{1}{s^{N-1}} \left[ \int_0^s \tau^{N-1} g(u) d\tau \right] ds$ and (2.12), by similar arguments there exists $m^* > 0$ such that if $\lambda \geq m^*$ then

$$v \left( \frac{1}{4} \right) \geq \mu \frac{\alpha_1 \alpha_2 B}{4} \left[ u \left( \frac{1}{4} \right) \right]^q.$$ (2.13)

Combining (2.11) and (2.13) we obtain

$$u \left( \frac{1}{4} \right) \geq \lambda \frac{\alpha_1 \alpha_2 A}{4} \left\{ \mu \frac{\alpha_1 \alpha_2 B}{4} \left[ u \left( \frac{1}{4} \right) \right]^q \right\}^p$$

which implies there exists a constant $\tilde{M}_3 > 0$ such that

$$u \left( \frac{1}{4} \right) \geq \tilde{M}_3 (\lambda \mu)^{\frac{1}{1-pq}}.$$ (2.14)

Similarly, there exists a constant $\tilde{M}_1 > 0$ such that

$$v \left( \frac{1}{4} \right) \geq \tilde{M}_1 (\mu \lambda q)^{\frac{1}{1-pq}}.$$ (2.15)
Now for $r > \frac{1}{4}$,

$$
-v'(r) = \frac{\mu}{r^{N-1}} \left[ \int_0^{\frac{r}{4}} s^{N-1} g(u) ds + \int_{\frac{r}{4}}^r s^{N-1} g(u) ds \right] \geq \frac{\mu}{r^{N-1}} \left[ g \left( u \left( \frac{1}{4} \right) \right) \left( \int_0^{\frac{1}{4}} s^{N-1} ds \right) + g(0) \right].
$$

Again by (2.12) and using arguments like those used before there exists a constant $m^{**}$ such that if $\lambda \geq m^{**}$ then

$$
-v'(r) \geq \mu \alpha_1 B \left[ u \left( \frac{1}{4} \right) \right]^q.
$$

Combining (2.14) and (2.16), there exists a constant $M_3^*$ such that

$$
-v'(r) \geq M_3^* \mu \lambda \left[ 1 - \frac{pq}{1 - pq} \right] \left( 1 - r \right) = M_3^* \left( \mu \lambda \right)^{\frac{1}{1 - pq}} \left( 1 - r \right).
$$

Integrating (2.17) from $r$ to 1 we obtain

$$
v(r) \geq M_3^* \left( \mu \lambda \right)^{\frac{1}{1 - pq}} \left( 1 - r \right) \quad \text{for } r \in \left( \frac{1}{4}, 1 \right).
$$

Now for $r \leq \frac{1}{4}$,

$$
v(r) \geq v \left( \frac{1}{4} \right) \geq \tilde{M}_1 \left( \mu \lambda \right)^{\frac{1}{1 - pq}} \left( 1 - r \right).
$$

Defining $M_3 = \min\{\tilde{M}_1, M_3^*\}$ we have

$$
v(r) \geq M_3 \left( \mu \lambda \right)^{\frac{1}{1 - pq}} \left( 1 - r \right) \quad \text{for } r \in (0, 1).
$$

Similarly, there exists a constant $M_1 > 0$ such that

$$
u(r) \geq M_1 \left( \mu \lambda \right)^{\frac{1}{1 - pq}} \left( 1 - r \right) \quad \text{for } r \in (0, 1).
$$

In particular, the L.H.S inequalities of Theorem 1.2 hold provided $\lambda, \mu$ are bigger than some constant $\tilde{M} > 0$. Next since

$$
u(r) = \lambda \int_r^1 \frac{1}{s^{N-1}} \int_0^s \tau^{N-1} f(v) d\tau ds \quad \text{and}
$$

$$
u(r) = \mu \int_r^1 \frac{1}{s^{N-1}} \int_0^s \tau^{N-1} g(u) d\tau ds
$$

we obtain $\|u\|_{\infty} \leq \lambda f(\|v\|_{\infty})$ and $\|v\|_{\infty} \leq \mu g(\|u\|_{\infty})$. Now by (2.15) w.l.o.g. we can assume that $\|v\|_{\infty}$ is large. Hence
∥v∥∞ \leq \mu g(∥u∥∞)
\leq \mu g(\lambda f(∥v∥∞))
\leq 2\mu B(\lambda f(∥v∥∞))^q
\leq 2\mu B\lambda^q[(2A∥v∥∞)^p]^q.

In particular, there exists a constant \( \tilde{M}_2 > 0 \) such that
\[
\|v\|_\infty \leq \tilde{M}_2(\mu^q (\lambda^q \mu))^{1-p/q} \tag{2.21}
\]

and
\[
-u'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(v) \, ds 
\leq \lambda f(\|v\|_\infty) 
\leq 2\lambda A\|v\|_p^p 
\leq 2\lambda A[\tilde{M}_2(\mu^q (\lambda^q \mu))^{1-p/q}]^p.
\]

Thus there exists a constant \( M_2 > 0 \) such that
\[
-u'(r) \leq M_2(\mu^p \lambda)^{1-p/q} \quad \text{for } r \in (0, 1)
\]

and hence integrating from \( r \) to \( 1 \) we obtain
\[
u(r) \leq M_2(\mu^p \lambda)^{1-p/q} (1 - r) \quad \text{for } r \in (0, 1).
\tag{2.22}
\]

Similarly, there exists \( M_4 \) such that
\[
v(r) \leq M_4(\lambda q \mu)^{1-p/q} (1 - r) \quad \text{for } r \in (0, 1).
\tag{2.23}
\]

and the R.H.S inequalities of Theorem 1.2 are established provided \( \lambda, \mu \) are bigger than some constant \( M^* > 0 \). Similarly, the result can be established in the case \( v\left(\frac{1}{2}\right) \geq \rho_2 \). Hence Theorem 1.2 is proven. \( \square \)

References