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An existence result on positive solutions for a class of p-Laplacian systems

D.D. Hai, R. Shivaji*

Department of Mathematics, Mississippi State University, Mississippi State, MS 39762, USA

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Abstract

Consider the system

$$-\Delta_p u = \lambda f(v) \quad \text{in } \Omega,$$

$$-\Delta_p v = \lambda g(u) \quad \text{in } \Omega,$$

$$u = v = 0 \quad \text{on } \partial\Omega,$$

where $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$, λ is a positive parameter, and Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$. We prove the existence of a large positive solution for λ large when

$$\lim_{x\to\infty} \frac{f(M(g(x)^{1/(p-1)})}{x^{p-1}} = 0$$

for every $M > 0$. In particular, we do not assume any sign conditions on $f(0)$ or $g(0)$.

Keywords: p-Laplacian systems; Semipositone; Positive solutions

1. Introduction

Consider the boundary value problem

$$\text{(I)} \quad \begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega, \\ -\Delta_p v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

* Corresponding author.
E-mail address: shivaji@ra.msstate.edu (R. Shivaji).

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where $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1, \lambda$ is a positive parameter, and Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$.

We are motivated by the paper of Dalmasso [3], in which system (I) was discussed when $p=2$ and f, g are increasing and $f, g \geqslant 0$. In [6], Hai–Shivaji extended the study of [3] applied to the case when no sign conditions on $f(0)$ or $g(0)$ were required, and without assuming monotonicity conditions on f or g. However, the results in [6] are for the semilinear case where proofs depend on the Green's functions. In this paper, we shall establish the existence results for system (I) with $p > 1$ under some additional assumptions than those required in [6]. In particular, our results apply to the case when $f(0)$ or $g(0)$ is negative, that is the semipositone case which is mathematically a challenging area in the study of positive solutions (see [1,7]). For a recent review on semipositone problems, see [2]. We refer to [5] for corresponding results in the single equation case. Our approach is based on the method of sub- and supersolutions (see e.g. [4]).

2. Existence results

We make the following assumptions:

(H.1) $f, g : [0,\infty) \to R$ are C^1, monotone functions such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty.$$

(H.2)

$$\lim_{x\to\infty} \frac{f(M(g(x)^{1/(p-1)}))}{x^{p-1}} = 0.$$

We shall establish:

Theorem A. *Let* (H.1), (H.2) *hold. Then there exists a positive number* λ^* *such that* (I) *has a large positive solution* (u, v) *for* $\lambda > \lambda^*$.

An example: *Let* $f(x)=\sum_{i=1}^m a_i x^{p_i} - C_1$, $g(x)=\sum_{j=1}^n b_j x^{q_j} - C_2$, where $p_i, q_j, a_i, b_i, C_1, C_2 > 0$ and $p_i q_j < (p-1)^2$. Then it is easy to see that f, g satisfy (H.1), (H.2).

Define $f(x) = f(0)$ and $g(x) = g(0)$ for $x < 0$. We shall establish Theorem A by constructing a positive weak subsolution (ψ_1, ψ_2) and supersolution (z_1, z_2) of (I) such that $\psi_i \leqslant z_i$ for $i = 1, 2$. That is, ψ_i satisfies

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1 \cdot \nabla q \,\mathrm{d}x \leqslant \lambda \int_\Omega f(\psi_2) q \,\mathrm{d}x,$$

$$\int_\Omega |\nabla\psi_2|^{p-2}\nabla\psi_2 \cdot \nabla q \,\mathrm{d}x \leqslant \lambda \int_\Omega g(\psi_1) q \,\mathrm{d}x,$$

and

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla q \,\mathrm{d}x \geqslant \lambda \int_\Omega f(z_2) q \,\mathrm{d}x$$

$$\int_\Omega |\nabla z_2|^{p-2}\nabla z_2 \cdot \nabla q \,\mathrm{d}x \geqslant \lambda \int_\Omega g(z_1) q \,\mathrm{d}x$$

for all $q \in H_0^1(\Omega)$ with $q \geqslant 0$.

Proof of Theorem A. Let λ_1 be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions and ϕ the corresponding positive eigenfunction with $\|\phi\|_\infty = 1$. Let $k_0, m, \delta > 0$ be such that $f(z), g(z) \geqslant -k_0$ for all $z \geqslant 0$ and $|\nabla\phi|^p - \lambda_1\phi^p \geqslant m$ on $\bar{\Omega}_\delta = \{x \in \Omega : \mathrm{d}(x, \partial\Omega) \leqslant \delta\}$. We shall verify that $(\psi_1, \psi_2) = (\psi, \psi)$, where $\psi = (\lambda k_0/m)^{1/(p-1)}((p-1)/p)\phi^{p/(p-1)}$, is a subsolution of (I) for λ large. Let $q \in H_0^1(\Omega)$ with $q \geqslant 0$. A calculation shows that

$$\begin{aligned}\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla q\,\mathrm{d}x &= \frac{\lambda k_0}{m}\int_\Omega \phi|\nabla\phi|^{p-2}\nabla\phi\cdot\nabla q\,\mathrm{d}x\\ &= \frac{\lambda k_0}{m}\left\{\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\nabla(\phi q)\,\mathrm{d}x - \int_\Omega |\nabla\phi|^p q\,\mathrm{d}x\right\}\\ &= \frac{\lambda k_0}{m}\int_\Omega (\lambda_1\phi^p - |\nabla\phi|^p)q\,\mathrm{d}x.\end{aligned}$$

Now, on $\bar{\Omega}_\delta$ we have $|\nabla\phi|^p - \lambda_1\phi^p \geqslant m$, which implies that

$$\frac{k_0}{m}(\lambda_1\phi^p - |\nabla\phi|^p) - f(\psi) \leqslant 0.$$

Next, on $\Omega \setminus \bar{\Omega}_\delta$ we have $\phi \geqslant \mu$ for some $\mu > 0$, and therefore for λ large,

$$f(\psi) \geqslant \frac{k_0}{m}\lambda_1 \geqslant \frac{k_0}{m}(\lambda_1\phi^p - |\nabla\phi|^p).$$

Hence

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla q\,\mathrm{d}x \leqslant \lambda\int_\Omega f(\psi)q\,\mathrm{d}x.$$

Similarly,

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla q\,\mathrm{d}x \leqslant \lambda\int_\Omega g(\psi)q\,\mathrm{d}x,$$

i.e. (ψ, ψ) is a subsolution of (I).

Next, let ϕ_0 be the solution of

$$-\Delta_p\phi_0 = 1 \quad \text{in } \Omega, \qquad \phi_0 = 0 \quad \text{on } \partial\Omega.$$

Let

$$(z_1, z_2) = \left(\frac{C}{\mu}\lambda^{1/(p-1)}\phi_0, [g(C\lambda^{1/(p-1)})]^{1/(p-1)}\lambda^{1/(p-1)}\phi_0\right),$$

where $\mu = \|\phi_0\|_\infty$ and $C > 0$ is a large number to be chosen later. We shall verify that (z_1, z_2) is a supersolution of (I) for λ large. To this end, let $q \in H_0^1(\Omega)$ with $q \geqslant 0$. Then we have

$$\begin{aligned}\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla q\,\mathrm{d}x &= \lambda\left(\frac{C}{\mu}\right)^{p-1}\int_\Omega |\nabla\phi_0|^{p-2}\nabla\phi_0\cdot\nabla q\,\mathrm{d}x\\ &= \lambda\left(\frac{C}{\mu}\right)^{p-1}\int_\Omega q\,\mathrm{d}x.\end{aligned}$$

By (H.2), we can choose C large enough so that

$$(C\lambda^{1/(p-1)})^{p-1} \geqslant (\mu^{p-1}\lambda) f([\lambda^{1/(p-1)}\mu][g(C\lambda^{1/(p-1)})]^{1/(p-1)}),$$

and therefore

$$\begin{aligned}\int_\Omega |\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla q \,\mathrm{d}x &\geqslant \lambda \int_\Omega f([\lambda^{1/(p-1)}\mu][g(C\lambda^{1/(p-1)})]^{1/(p-1)}) q \,\mathrm{d}x \\ &\geqslant \lambda \int_\Omega f(z_2) q \,\mathrm{d}x.\end{aligned}$$

Next,

$$\begin{aligned}\int_\Omega |\nabla z_2|^{p-2}\nabla z_2 \cdot \nabla q \,\mathrm{d}x &= \lambda g(C\lambda^{1/(p-1)}) \int_\Omega q \,\mathrm{d}x \\ &\geqslant \lambda \int_\Omega g(C\mu^{-1}\lambda^{1/(p-1)}\phi_0) q \,\mathrm{d}x \\ &= \lambda \int_\Omega g(z_1) q \,\mathrm{d}x,\end{aligned}$$

i.e. (z_1, z_2) is a supersolution of (I) with $z_i \geqslant \psi$ for C large, $i = 1, 2$. Thus, there exists a solution (u, v) of (I) with $\psi \leqslant u \leqslant z_1$, $\psi \leqslant v \leqslant z_2$. This completes the proof of Theorem A. □

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