GLOBAL BIFURCATION OF POSITIVE SOLUTIONS IN SOME SYSTEMS OF ELLIPTIC EQUATIONS*

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Abstract. In this paper the structure of the nonnegative steady-state solutions of a system of reaction-diffusion equations arising in ecology is investigated. The equations model a situation in which a predator species and a prey species inhabit the same region and the interaction terms are of Holling–Tanner type so that the predator has finite appetite. Prey and predator birth-rates are treated as bifurcation parameters and the theorems of global bifurcation theory are adapted so that they apply easily to the system. Thus ranges of parameters are found for which there exist nontrivial steady-state solutions.

Key words. reaction-diffusion equations, multiple steady states, global bifurcation, predator-prey

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1. Introduction. In this paper we study the nonnegative steady-state solutions of the reaction-diffusion system

\begin{align}
  u_t(x,t) - d_1 \Delta u(x,t) &= au - a_1 u^2 - a_2 uv/(1 + mu), \\
  v_t(x,t) - d_2 \Delta v(x,t) &= bv - b_1 v^2 + b_2 uv/(1 + mu)
\end{align}

for \( x \in D \) and \( t \geq 0 \) where \( D \) is a bounded region in \( \mathbb{R}^n \) \((n = 1, 2, 3)\) with smooth boundary together with boundary conditions

\begin{align}
  u(x,t) = 0 = v(x,t) \quad \text{for all} \quad x \in \partial D \quad \text{and} \quad t \geq 0.
\end{align}

Equation (1.1) models a situation in which a prey and a predator species with population densities \( u(x,t) \) and \( v(x,t) \) respectively inhabit the region \( D \). It is assumed that both species diffuse, i.e., move from points of high to points of low population density; the Laplacian terms in (1.1) correspond to this diffusion, the constants \( d_1 \) and \( d_2 \) giving the rates at which the species diffuse. It is also assumed that in the absence of other species and of diffusion that both species would grow logistically. Thus, in the absence of other factors, the rate of increase of the prey population is given by \( au - a_1 u^2 \). If \( u \) is small, this increase is approximately equal to \( au \) and the constant \( a \) is termed the birth rate of the prey. Because of limited natural resources, the prey population will decrease in size if it becomes too large; we assume throughout that the constant \( a_1 > 0 \) so that \( au - a_1 u^2 < 0 \) for sufficiently large \( u \). Similarly the constant \( b \) is termed the birth rate of \( v \) and we assume that the constant \( b_1 > 0 \). The term \( a_2 uv/(1 + mu) \) represents the rate at which the prey is consumed by the predator and is usually referred to as the Holling–Tanner interaction term; as is reasonable this term increases as either \( u \) or \( v \) increases. In the classical equations of ecology the corresponding term is simply \( a_2 uv \). This classical interaction term has the defect that for a fixed predator population \( \lim_{u \to \infty} a_2 uv = \infty \) which implies that predators must be capable of consuming prey at an infinitely great rate. For the Holling–Tanner interaction term, however, \( \lim_{u \to \infty} a_2 uv/(1 + mu) = a_2 v/m \) and this difficulty does not arise. We assume throughout that the constants \( a_2, b_2 > 0 \).

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In [2] we used a decoupling technique to study the steady-state solutions of the classical equations of ecology, viz.

\begin{align*}
-d_1 \Delta u &= au - a_1 u^2 - a_2 u w, \\
-d_2 \Delta v &= bv - b_1 v^2 + b_2 w
\end{align*}

with Dirichlet boundary conditions. The general idea is to regard \( v \) as a fixed function in the first equation in (1.3) and solve for \( u \), denoting the solution by \( u(v) \), i.e., \( u(v) \) is the solution of

\begin{align*}
-d_1 \Delta u + a_2 u w &= au - a_1 u^2 & \text{on } D, \\
 u|_{\partial D} &= 0
\end{align*}

The solutions of (1.4) are easy to describe and a suitable \( u(v) \) can be defined. Then \( u(v) \) is substituted into the second equation in (1.3) to give a single equation for \( v \). Treating \( b \) as a bifurcation parameter and using the results of global bifurcation theory, fairly detailed results are obtained about the solutions of (1.3).

If we fix \( v \) in the first equation with the Holling–Tanner interaction term, it seems considerably harder to analyze the solutions of the corresponding equation for \( u \), viz.,

\begin{align*}
-d_1 \Delta u &= au - a_1 u^2 - \frac{a_2 w}{1 + mu} & \text{on } D, \\
 u|_{\partial D} &= 0
\end{align*}

and we are no longer able to use our decoupling technique. In this paper we study nontrivial steady-state solutions of a simplified version of (1.1), i.e.,

\begin{align*}
-\Delta u &= au - a_1 u^2 - a_2 w(1 + mu) & \text{on } D, \\
 u|_{\partial D} &= 0, \\
-\Delta v &= bv - b_1 v^2 + b_2 w(1 + mu) & \text{on } D, \\
v|_{\partial D} &= 0
\end{align*}

by applying bifurcation theory directly to the system (1.5). Note that we assume \( d_1 = d_2 = 1 \); this is done simply for notational convenience and it is straightforward to adapt our proofs to deal also with the general case of unequal diffusion coefficients. If we assume that all the other constants in (1.5) are fixed and treat \( a \) as a bifurcation parameter, we can show that (1.5) is equivalent to an operator equation of the form \( w = T(a, w) = 0 \) where \( w = (u, v) \).

In the existing literature global bifurcation theorems of the type we require seem only to apply in the cases where the Fréchet derivative \( T_w(a, 0) = a A w \) (see Rabinowitz [11]) or \( T_w(a, 0) = a A w + A_2 w \) (see Chow and Hale [4]) where \( A, A_1 \) and \( A_2 \) are linear operators. In some situations which we encounter \( T_w(a, 0) \) depends on \( a \) in a more complicated way and in §3 we give a formulation of some standard theorems on bifurcation which can be applied easily to all the cases in which we are interested. In §2 we discuss results we shall require later on linear problems and on the trivial solutions of (1.5). In §§4 and 5 we treat \( b \) and \( a \) respectively as bifurcation parameters.

We work throughout with Dirichlet boundary conditions. Our results also apply to the more ecologically reasonable cases of Neumann and Robin boundary conditions. However Dirichlet boundary conditions present the hardest mathematical problem and so we concentrate our attention on these. In the case of Neumann boundary conditions all the steady-state solutions we obtain are spatially homogeneous.

A number of other studies have been made on the existence of steady-state solutions of the classical equations of ecology. In Dancer [7] index theory is used to give necessary and sufficient conditions for the existence of nontrivial solutions. Leung [9] has obtained existence and uniqueness results by using iteration methods. Local bifurcation methods for the classical equations describing competing species are used in Cantrell and Cosner [3].
2. Preliminaries. It is well known that the linear eigenvalue problem
\[-\Delta \phi = \lambda \phi \quad \text{on } D, \quad \phi = 0 \quad \text{on } \partial D\]
has an infinite sequence of eigenvalues \( \{\lambda_n\} \) such that \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) with corresponding eigenfunctions \( \phi_1, \phi_2, \phi_3, \cdots \) where \( \phi_1(x) > 0 \) for \( x \in D \). Suppose that \( q: D \to \mathbb{R} \) is smooth. Then the linear eigenvalue problem
\[(2.1) \quad -\Delta u + qu = \lambda u \quad \text{on } D, \quad u = 0 \quad \text{on } \partial D\]
also has an infinite sequence of eigenvalues which are bounded below. We denote the \( i \)th eigenvalue of (2.1) by \( \lambda_i(q) \). It is known that \( \lambda_1(q) \) is a simple eigenvalue and that the corresponding eigenfunctions do not change sign on \( D \). Clearly \( \lambda_1(0) = \lambda_1 \) and \( \lambda_1(q) \) is an increasing function of \( q \).

If \( 0 \) is not an eigenvalue of (2.1), then we can define a corresponding solution operation \( K \), i.e., \( Kf \) is the unique solution of
\[-\Delta u + qu = f \quad \text{on } D, \quad u = 0 \quad \text{on } \partial D\]
i.e., \( K \) is the inverse of the differential operator \( L = -\Delta + q \) associated with Dirichlet boundary conditions. It is well known (see e.g. Amann [1] and Sattinger [12]) that \( K: C^1(D) \to C^1(D) \) and \( K: L_2(D) \to L_2(D) \) is a compact operator and that \( K: C^{2+\alpha}(D) \to C_0^{2+\alpha}(D) = \{ u \in C^{2+\alpha}(D) : u(x) = 0 \text{ for } x \in \partial D \} \).

Consider now the nonlinear boundary value problem
\[(2.2) \quad -\Delta u + qu = au - a_1 u^2 \quad \text{on } D, \quad u = 0 \quad \text{on } \partial D\]
where \( q \) is as above and \( a \) and \( a_1 \) are real numbers with \( a_1 > 0 \). It is known that if \( a \leq \lambda_1(q) \) then \( u = 0 \) is the only nonnegative solution of (2.2) whereas if \( a > \lambda_1(q) \) then (2.2) has a solution \( u \) which is positive on \( D \). Since \( u \to (au - a_1 u^2)/u \) is a decreasing function, it follows that (see Cohen and Laetsch [5]) for each fixed \( a > \lambda_1(q) \) there is a unique solution of (2.2) which is positive on \( D \).

Suppose now that \( q = 0 \). We denote the unique positive solution of
\[(2.3) \quad -\Delta u = au - a_1 u^2 \quad \text{on } D, \quad u|_{\partial D} = 0\]
where \( a > \lambda_1 \) by \( u_a \). In the \((a,u)\) plane, i.e., \( \mathbb{R} \times C^1(D) \), the curve of solutions \( a \to u_a \) bifurcates from the zero solution when \( a = \lambda_1 \). The linearized operator corresponding to \( u_a \) is the differential operator \( L \) where
\[Lu(x) = -\Delta u(x) - au(x) + 2a_1 u_a(x) u(x)\]
associated with Dirichlet boundary conditions.

**Lemma 2.1.** All eigenvalues of \( L \) are strictly positive.

*Proof.* Since
\[(2.4) \quad -\Delta u_a + (a_1 u_a - a) u_a = 0 \quad \text{on } D, \quad u_a|_{\partial D} = 0,\]
\( u_a \) is a positive eigenfunction of \( -\Delta + (a_1 u_a - a) \) corresponding to the eigenvalue \( 0 \) and so \( \lambda_1(a_1 u_a - a) = 0 \). Hence \( \lambda_1(2a_1 u_a - a) > 0 \) and the result is proved.

We now show that \( u_a \) depends continuously on \( a \). Define \( F: (\lambda_1, \infty) \times C_0^{2+\alpha}(D) \to C^{\alpha}(D) \) by
\[F(s,u) = -\Delta u - su + a_1 u^2.\]
Clearly $F$ is a $C^1$ function and $F(s, u) = 0$ for all $s > \lambda_1$. Choose and fix $a > \lambda_1$. If we denote the Fréchet derivative $F'(a, u)$ by $L$, then $L: C^{2+\alpha}_0(D) \to C^\alpha(D)$ such that $Lu = -\Delta u - au + 2a_1u$. By Lemma 2.1 $L$ is an isomorphism and so by the Implicit Function Theorem there exists a $C^1$ function $\phi: \mathbb{R} \to C^{2+\alpha}_0(D)$ defined on a neighbourhood of $a$ such that $\phi(a) = u_a$ and $F(s, \phi(s)) = 0$. However, $F(s, u) = 0$ has the unique solution $(s, u_s)$ close to $(a, u_a)$ and so $\phi(s) = u_s$. This $s \to u_s$ is a $C^1$ map from $\mathbb{R}$ to $C^{2+\alpha}_0(D)$.

Let $\eta_a = du_a/da$. Differentiation of (2.4) with respect to $a$ and interchange of order of the smooth derivatives involved show that

$$-\Delta \eta_a + (2a_1u_a - a) \eta_a = u_a > 0 \quad \text{on } D,$$

i.e.

$$L\eta_a > 0 \quad \text{on } D, \quad \eta_a|_{\partial D} = 0.$$

Since by Lemma 2.1 the principal eigenvalue of $L$ on $D$ is positive, there exists a region $\bar{D}$ containing $D$ such that the principal eigenvalue of $L$ with Dirichlet boundary conditions on $\bar{D}$ is positive; the corresponding principal eigenfunction $\phi$ is such that $\phi$ and $L\phi$ are strictly positive on $\bar{D}$. Hence it follows from the generalized maximum principle (see [10, Chap. 2, Thm. 10]) that $\eta_a/\phi$ does not have a nonpositive minimum in $D$ and so $\eta_a > 0$. Thus we have

**Lemma 2.2.** The map $a \to u_a$ is a $C^1$ map from $(\lambda_1, \infty) \to C^{2+\alpha}_0(D)$ and, if $\eta_a = du_a/da$, then $\eta_a(x) > 0$ for all $x$ in $D$.

In a similar way we can define and establish the corresponding properties of $v_b$, the unique positive solution of

$$-\Delta v = bv - b_1v^2 \quad \text{on } D, \quad v|_{\partial D} = 0$$

when $b > \lambda_1$.

We now discuss the trivial solutions of (1.5). Clearly for all values of $a$ and $b$ there is the zero solution i.e. $u = 0$ and $v = 0$. When $a > \lambda_1$, there is the semi-trivial solution $u = u_a$ and $v = 0$ and, when $b > \lambda_1$, there is the semi-trivial solution $u = 0$ and $v = v_b$. We prove the existence of nontrivial solutions by studying the bifurcations which occur from branches of semi-trivial solutions. For this purpose it is necessary to obtain some a priori information about solutions of (1.5).

**Lemma 2.3.** If $a > \lambda_1$, then $(a - \lambda_1)\phi/a_1 \leq u_a \leq a/a_1$ where $\phi$ is the principal eigenfunction of $-\Delta$ such that $\max \phi = 1$.

**Proof.** It is easy to check that $(a - \lambda_1)\phi/a_1$ and $a/a_1$ are sub and supersolutions of (2.3). But $u_a$ is the unique positive solution of (2.3) and so must lie between the sub- and supersolution.

**Lemma 2.4.** If $(u, v)$ is a nonnegative solution of (1.5) such that $u$ is not identically zero, then $a > \lambda_1$.

**Proof.** Since $u$ satisfies the first equation in (1.5), it follows that $-\Delta u < au$ on $D$. Multiplying by $u$ and integrating over $D$ shows that $\int_D |\nabla u|^2 dx < a\int_D u^2 dx$. But by Poincaré's Inequality $\int_D |\nabla u|^2 dx \leq \lambda_1 \int_D u^2 dx$ and so $a > \lambda_1$.

The above lemma shows that the prey cannot coexist with the predator if its birth rate is too low. The next lemma gives a priori bounds on the population densities in terms of the birth rates.

**Lemma 2.5.** Suppose $(u, v)$ is a nonnegative solution of (1.5) such that $u \not\equiv 0$ and $v \not\equiv 0$. Then

(i) $u \leq u_a$ and $v \leq b_1^{-1}[b + b_2a/(a_1 + ma)]$;

(ii) if $b > \lambda_1$, then $v \geq v_b$. 
Proof. (i) Since \( u \neq 0 \), it follows that \( a > \lambda_1 \) and so (2.3) has the unique positive solution \( u_a \). Clearly \( u \) is a subsolution of (2.3) and there exist arbitrarily large supersolutions of (2.3). Hence \( u_a \) must be greater than or equal to the subsolution \( u \).

As \( v \) satisfies the equation

\[
-\Delta v = \left[ b - b_i v + b_2 u/(1 + mu) \right] v \quad \text{on } D
\]

and \( u \leq u_a \leq a/a_1 \), we have that

\[
-\Delta v \leq \left[ b - b_i v + ab_2/(a_1 + ma) \right] v
\]

and so \( \Delta v > 0 \) whenever \( v > b_i^{-1}[b + b_2 a/(a_1 + ma)] \) (\( = B \) say.) Thus it is impossible that \( v \) has a local maximum at \( x_0 \) where \( v(x_0) > B \) and so we obtain the required upper bound for \( v \).

(ii) Regarding \( u \) as a fixed function, \( v \) is the unique positive solution of (2.5) with Dirichlet boundary conditions. Clearly \( v_b \) is a subsolution of (2.5) and, as there are arbitrarily large constant supersolutions of (2.5), it follows that \( v \geq v_b \).

Finally in this section we make a preliminary investigation of bifurcation from the branch of trivial solutions of the form \( u = u_a, v = 0 \). Writing \( u = u_a - U \) and \( v = V \), it is easy to check that \( (u, v) \) is a nonnegative solution of (1.5) if and only if \( 0 \leq U \leq u_a, V \geq 0 \) and \( (U, V) \) satisfies

\[
-\Delta U = aU - 2a_1 u_a U + a_2 u_a V/(1 + mu_a) + f(a, x, U, V),
-\Delta V = bV + b_2 u_a V/(1 + mu_a) + g(a, x, U, V)
\]

where \( f \) and \( g \) are smooth functions on \( [\lambda_1, \infty) \times D \times \mathbb{R} \times \mathbb{R} \) such that

\[
f(a, x, U, V) = a_1 U^2 + a_2 \left[ (u_a - U) V/(1 + m(u_a - U)) - u_a V/(1 + mu_a) \right],
\]

\[
g(a, x, U, V) = -b_1 V^2 + b_2 \left[ (u_a - U) V/(1 + m(u_a - U)) - u_a V/(1 + mu_a) \right]
\]

for \( U \leq u_a(x) + \frac{1}{2} m^{-1} \). Let \( F: (\lambda_1, \infty) \times C^1(D) \times C^1(D) \to C^1(D) \) be defined by

\[
F(a, U, V)(x) = f(a, x, U(x), V(x))
\]

and let \( G \) be the similar operator corresponding to \( g \). Clearly \( F \) and \( G \) are continuous and the Fréchet derivatives \( F_{(U,V)}(a,0,0) \) and \( G_{(U,V)}(a,0,0) \) are zero. Then equation (2.6) can be written as

\[
U = aKU - 2a_1 K(u_a U) + a_2 K \left[ u_a V/(1 + mu_a) \right] + KF(a, U, V),
V = bKV + b_2 K \left[ u_a V/(1 + mu_a) \right] + KG(a, U, V)
\]

where \( K \) is the inverse of \( -\Delta \) with Dirichlet boundary conditions. Clearly (2.7) has the solution \( U = 0, V = 0 \) for all values of \( a \) and \( b \) and to find possible bifurcation points from the branch of zero solutions it is necessary to investigate the linearisation

\[
U = aKU - 2a_1 K(u_a U) + a_2 K \left[ u_a V/(1 + mu_a) \right],
V = bKV + b_2 K \left[ u_a V/(1 + mu_a) \right].
\]

3. Bifurcation theory. Since the dependence of the linearisation (2.8) on \( a \) and \( b \) is quite complicated, it is necessary to reformulate some of the standard theorems of bifurcation before we can apply them to our equations.
Let $X$ be a Banach space and let $T: \mathbb{R} \times X \to X$ be a compact, continuously differentiable operator such that $T(a,0)=0$. Suppose we can write $T$ as

$$T(a,u) = K(a)u + R(a,u)$$

where $K(a)$ is a linear compact operator and the Fréchet derivative $R_u(a,0)=0$. We investigate bifurcation phenomena for the equation

$$u = T(a,u)$$

treating $a$ as a bifurcation parameter.

**Theorem 3.1.** Suppose $(a,0)$ is a bifurcation point of (3.2). Then

(i) $K(a)$ has eigenvalue 1;

(ii) if $\{(a_n,u_n)\}$ is a sequence of nontrivial solutions such that $a_n \to a$ and $u_n \to 0$, then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/\|u_n\| \to u_0$ where $u_0$ is an eigenvector of $K(a)$ corresponding to the eigenvalue 1.

**Proof.** Since $u_n - T(a_n,u_n)=0$ for all $n$, we have that

$$u_n - T_u(a_n,0)u_n = \int_0^1 [T_u(a_n,su_n)u_n - T_u(a_n,0)u_n] \, ds.$$ 

Let $v_n = u_n/\|u_n\|$. Then

$$v_n - K(a)v_n = \int_0^1 [T_u(a_n,su_n)v_n - T_u(a_n,0)v_n] \, ds.$$ 

As $n \to \infty$, the integral term $\to 0$ and so $\lim_{n \to \infty} (v_n - K(a)v_n)=0$. Since $\{v_n\}$ is bounded and $K(a)$ is compact, there exists a subsequence such that $\{K(a)v_n\}$ is convergent. Hence there exists a subsequence of $\{v_n\}$ converging to $u_0$ say. As $\|v_n\|=1$ for all $n$, $u_0 \neq 0$ and clearly $u_0 - K(a)u_0=0$.

We now give a sufficient condition for bifurcation to occur and at the same time obtain a global bifurcation result. First we recall the notions of multiplicity of an eigenvalue and the index of a fixed point.

Let $K: X \to X$ be a compact linear operator and let $\lambda_0$ be a nonzero eigenvalue of $K$. Then the null space of $K-\lambda_0 I$ denoted by $N(K-\lambda_0 I)$ is nonempty. We define the generalized null space of $\lambda_0$ as $M(K; \lambda_0)=\bigcup_{p=1}^{\infty} N(K-\lambda_0 I)^p$. It is well known that

$$N(K-\lambda_0 I) \subset N(K-\lambda_0 I)^2 \subset N(K-\lambda_0 I)^3 \subset \cdots$$

and that there exists $P$ such that $N(K-\lambda_0 I)^p \subset N(K-\lambda_0 I)^{p+1}$ for $p < P$ but $N(K-\lambda_0 I)^P = N(K-\lambda_0 I)^{P+1} = N(K-\lambda_0 I)^{P+2} = \cdots$ and so

$$M(K; \lambda_0) = \bigcup_{p=1}^{P} N(K-\lambda_0 I)^p.$$ 

Thus $\dim M(K; \lambda_0) < \infty$ and we define the algebraic multiplicity of $\lambda_0$ as equal to $\dim M(K; \lambda_0)$. If $\lambda_0$ has algebraic multiplicity 1, we say that $\lambda_0$ is a simple eigenvalue. Clearly $\lambda_0$ is a simple eigenvalue if and only if $\dim N(K-\lambda_0 I) = \dim N(K-\lambda_0 I)^2 = 1$. It is easy to show that $N(K-\lambda_0 I) = N(K-\lambda_0 I)^2$ if and only if $N(K-\lambda_0 I) \cap R(K-\lambda_0 I) = \{0\}$ where $R(K-\lambda_0 I)$ denotes the range space of $K-\lambda_0 I$. Thus $\lambda_0$ is a simple eigenvalue of $K$ if and only if $\dim N(K-\lambda_0 I) = 1$ and $N(K-\lambda_0 I) \cap R(K-\lambda_0 I) = \{0\}$. 

Suppose now that \( I - K : X \to X \) is a bijection. Then it is well known that the Leray-Schauder degree \( \deg(I - K, B, 0) = (-1)^p \) where \( B \) is a ball centre 0 in \( X \) and \( p = \sum \) of the algebraic multiplicities of the eigenvalues of \( K \) which are \( > 1 \) (see Krasnoselskii [8]). Suppose \( T : X \to X \) is a compact differentiable operator. If \( x_0 \) is an isolated fixed point of \( T \), we define the index of \( T \) at \( x_0 \) as \( \alpha(T, x_0) = \deg(I - T, B, x_0) \) where \( B \) is a ball centre \( x_0 \) such that \( x_0 \) is the only fixed point of \( T \) in \( B \). If \( x_0 \) is a fixed point of \( T \) such that \( I - T'(x_0) \) is invertible, then \( x_0 \) is an isolated fixed point of \( T \) and

\[
i(T, x_0) = \deg(I - T, B, x_0) = \deg(I - T'(x_0), \hat{B}, 0)
\]

where \( B \) is a sufficiently small ball centre \( x_0 \) and \( \hat{B} \) is a ball centre 0.

We now state the result we shall use on global bifurcation. Suppose \( T : \mathbb{R} \times X \to X \) is as given by (3.1).

**Theorem 3.2.** Let \( a_0 \) be such that \( I - K(a) \) is invertible if \( 0 < |a - a_0| < \varepsilon \) for some \( \varepsilon > 0 \). Suppose \( \alpha(T(a, \cdot), 0) \) is constant on \( (a_0 - \varepsilon, a_0) \) and on \( (a_0, a_0 + \varepsilon) \) such that, if \( a_0 - \varepsilon < a_1 < a_0 < a_2 < a_0 + \varepsilon \), then \( \alpha(T(a_1, \cdot), 0) \neq \alpha(T(a_2, \cdot), 0) \). Then there exists a continuum \( C \) in the \((a - u)\)-plane of solutions of (3.2) such that one of the following alternatives holds

(i) \( C \) joins \((a_0, 0)\) to \((a_1, 0)\) where \( I - K(\hat{a}) \) is not invertible.

(ii) \( C \) joins \((a_0, 0)\) to \( \infty \) in \( \mathbb{R} \times X \).

The above results can be proved by using exactly the same argument as in Rabinowitz [11]. The index \( \alpha(T(a, \cdot), 0) \) can be calculated by investigating the eigenvalues of \( K(a) \).

4. **Structure of solutions with \( b \) as bifurcation parameter.** In this section we shall regard \( b \) as a bifurcation parameter and suppose that all other constants are fixed. For all values of \( b \) we have the branch of zero solutions of (1.5) \( S_0 = \{(b, 0, 0) : b \in \mathbb{R}\} \). When \( b \) crosses \( \lambda_1 \), there bifurcates from \( S_0 \) the branch of semi-trivial solutions \( S_1 = \{(b, 0, \psi) : b > \lambda_1 \} \). Lemma 2.4 shows that, when \( a \) is fixed \( < \lambda_1 \), then all nonnegative solutions of (1.5) lie on either \( S_0 \) or \( S_1 \). For the rest of this section we suppose that \( a \) is fixed \( > \lambda_1 \) so that we also have the branch of semi-trivial solutions \( S_2 = \{(b, u, \nu) : b \in \mathbb{R}\} \). We show that there is a continuum of nontrivial solutions (i.e. in which neither \( u \) nor \( v \) is identically zero) joining \( S_1 \) and \( S_2 \).

First we use the result of Crandall and Rabinowitz [6] on bifurcation from a simple eigenvalue to obtain a local result on bifurcation from \( S_2 \). Motivated by (2.7), we define \( T : \mathbb{R} \times C^1(D) \times C^1(D) \to C^1(D) \times C^1(D) \) by (1.5).

For notational convenience let \( q(x) = u_a(x)/(1 + mu_a(x)) \). Let \( b_0 = \lambda_1(-b_2q) \) and let \( \psi_1 \) denote a corresponding nonnegative principal eigenfunction corresponding to principal eigenvalue \( b_0 \) of

\[
- \Delta \psi - b_2q \psi = b \psi \quad \text{on } D, \quad \psi|_{\partial D} = 0.
\]
It is easy to check that $N(H_{(u,v)}(b_0,0,0)) = \text{span} \{ (\phi_1, \psi_1) \}$ where $\phi_1 = a_2 K_1 [q \psi_1]$ and $K_1$ denotes the inverse of $-\Delta - a + 2a_1 u_a$ with Dirichlet boundary conditions. Thus $\dim N(H_{(u,v)}(b_0,0,0)) = 1$. It follows from the properties of compact operators that the codimension of $R(H_{(u,v)}(b_0,0,0)) = 1$.

Using (4.1), it is easy to see that the Fréchet derivative $$H_{(u,v)}(b_0,0,0)(\phi_1, \psi_1) = (0, -K \psi_1).$$

Suppose that $(0, -K \psi_1) \in R(H_{(u,v)}(b_0,0,0))$. Then there exists $\psi \in C^1(D)$ such that $$\psi - bK \psi - b_2 K [q \psi] = -K \psi_1$$

and so

\begin{equation}
-\Delta \psi - b \psi - b_2 q \psi = -\psi_1.
\end{equation}

Multiplying (4.2) by $\psi_1$ and integrating over $D$ shows that $\int_D \psi_1^2 dx = 0$ and this is impossible. Thus $(0, -K \psi_1) \notin R(H_{(u,v)}(b_0,0,0))$.

We have shown that $H$ satisfies all the hypotheses of Theorem 1.7 of [6]. Thus there exists a real interval $(-\epsilon, \epsilon)$ and functions $b: (-\epsilon, \epsilon) \to \mathbb{R}$, $u, v: (-\epsilon, \epsilon) \to C^1(D)$ such that the nontrivial zeros of $H$ close to $(b_0,0,0)$ lie on the curve $\{(b(s), s \phi_1 + su(s), s\psi_1 + sv(s)) : -\epsilon < s < \epsilon\}$ where $b(0) = b_0$, $u(0) = v(0) = 0$. It follows that for the system of equations (1.5) bifurcation occurs from the branch of semi-trivial solutions $S_2$ at $(b_0, u_a, 0)$ and close to the bifurcation point the nontrivial solutions lie on the curve $\{(b(s), u_a - s \phi_1 - su(s), s\psi_1 + sv(s)) : -\epsilon < s < \epsilon\}$. Points on the curve with $s > 0$ correspond to nontrivial, nonnegative solutions of (1.5).

We now investigate the global nature of the above curve of nontrivial, nonnegative solutions in the $b-(u,v)$ plane, i.e., in $\mathbb{R} \times C^1(D) \times C^1(D)$. Theorems 4.1 and 4.2 give limitations on the values of $b$ for which such solutions can exist.

**Theorem 4.1.** If $(b, u, v)$ be a nontrivial, nonnegative solution of (1.5), then $b > b_0$ (i.e. $b > \lambda_1(-b_2 q)$).

**Proof.** We have that

$$-\Delta v - [b_2 u/(1 + mu)] v = bv - b_1 v^2.$$  

Since $u \leq u_a$, it follows that

$$-\Delta v - b_2 q v \leq bv - b_1 v^2.$$  

Multiplying by $v$ and integrating over $D$, we obtain

$$-\int_D (\Delta v + b_2 q v) v \, dx < b \int_D v^2 \, dx.$$  

By the spectral theorem

$$-\int_D (\Delta v + b_2 q v) v \, dx \leq b_0 \int_D v^2 \, dx$$

and so it follows that $b > b_0$.

The above result shows that the bifurcation of nonnegative solutions from $S_2$ at $(b_0, u_a, 0)$ must be to the right. The next result shows that the branch of nontrivial solutions cannot extend too far to the right.

**Theorem 4.2.** There exists $M > 0$ such that, if $(b, u, v)$ is a nontrivial, nonnegative solution of (1.5), then $b \leq M$.  


Proof. By Lemma 2.3 \( v_b \geq (b-\lambda_1)\phi/b_1 \) where \( \phi \) is the principal eigenfunction of \( -\Delta \) with \( \sup \phi = 1 \). If \( \alpha \) is a positive constant, it follows that \( \lim_{b \to \infty} \lambda_1(\alpha v_b) = \infty \) (see [2, Thm. 3.4]). Choose \( M \) such that \( \lambda_1(a_1 a_2 v_b/(a_1 + ma)) > a \) whenever \( b > M \).

Suppose \((b,u,v)\) is a nontrivial, nonnegative solution of (1.5). Then \( u \) is a nontrivial solution of

\[-\Delta u + \left[ a_2 v/(1 + mu) \right] u = au - a_1 u^2 \]

and so \( a > \lambda_1(a_2 v/(1 + mu)) \). Since \( v \leq v_b \) and \( u \leq a/a_1 \), it follows that \( a > \lambda_1(a_1 a_2 v_b/(a_1 + ma)) \). Hence we must have that \( b < M \).

We now show that \( T \) satisfies the hypotheses of our global bifurcation result, i.e., Theorem 3.2. We can write \( T \) as

\[ T(b,u,v) = K(b)(u,v) + R(b,u,v) \]

where \( K(b) \) is the compact linear operator such that \( K(b)(u,v) = (aKu - 2a_1 K(u, u) + a_2 K(v, v), bKv + b_2 K(v, v)) \) and \( K(b) \) and \( R(b,u,v) \) satisfy the conditions given at the start of \( \S3.1 \). In order to show that the hypotheses of Theorem 3.2 are satisfied, we must calculate the index \( i(T(b, \cdot), 0) \) when \( b \) is close to \( b_0 \). This index is equal to \(-1/b\) where \( b \) is the sum of the algebraic multiplicities of eigenvalues of \( K(b) > 1 \).

Suppose that \( \mu > 0 \) is an eigenvalue of \( K(b) \). Then there exists a nonzero function \( v \) such that

\[ bKv + b_2 K(qv) = \mu v \]

and so

\[-\mu \Delta v - b_2 qv = bv \text{ on } D, \quad v = 0 \text{ on } \partial D, \]

i.e., \( b \) is an eigenvalue of (4.3). Conversely, if \( \mu \geq 1 \) and \( b \) is an eigenvalue of (4.3) with corresponding eigenfunction \( v \), then \((u,v)\) is an eigenfunction of \( K(b) \) corresponding to the eigenvalue \( \mu \) where \( u \) is the unique solution of

\[-\Delta u - au + 2a_1 u u = a_2 qv \text{ on } D, \quad u = 0 \text{ on } \partial D. \]

Note that, since all eigenvalues of \(-\Delta - a + 2a_1 u u \) are positive by Lemma 2.1 and \( \mu > 1 \), it follows that \(-\mu \Delta - a + 2a_1 u u \) is invertible. The eigenvalues of (4.3) form an increasing sequence \( \gamma_i(\mu) < \gamma_2(\mu) \leq \gamma_3(\mu) \leq \cdots \) and the variational characterisation of eigenvalues shows that \( \mu \to \gamma_i(\mu) \) is a continuous increasing function. Thus \( \mu \geq 1 \) is an eigenvalue of \( K(b) \) if and only if \( b = \gamma_i(\mu) \) for some \( \mu \). Clearly \( \gamma_i(1) = \lambda_i(-b_2 q) \).

Suppose that \( b < b_0 \), i.e., \( b < \lambda_1(-b_2 q) \). Hence \( b < \gamma_i(1) \) and so \( b < \gamma_i(\mu) \) for \( i = 1,2, \cdots \) and \( \mu \geq 1 \). Hence \( K(b) \) has no eigenvalues \( > 1 \) and so \( i(T(b, \cdot), 0) = 1 \).

Suppose \( b_0 < b < \lambda_2(-b_2 q) \). Then \( \gamma_i(1) < b < \gamma_i(\mu) \). Since \( \mu \to \gamma_i(\mu) \) is increasing with \( \lim_{\mu \to \infty} \gamma_i(\mu) = \infty \), there exists a unique \( \mu > 1 \) \( (\mu_1 \text{ say}) \) such that \( b = \gamma_i(\mu_1) \). Since \( b < \gamma_2(1) \), it follows that \( b < \gamma_i(\mu) \) for \( i = 2,3, \cdots \) and \( \mu \geq 1 \). Thus \( \mu_1 \) is the only eigenvalue of \( K(b) \) which is greater than \( 1 \). We now show that \( \mu_1 \) is a simple eigenvalue of \( K(b) \). The discussion above shows that \( N(K - \mu_1 I) = \text{span}\{\phi, \psi\} \) where \( \psi \) is the principal eigenfunction corresponding to the eigenvalue \( b \) of

\[-\mu_1 \Delta v - b_2 qv = bv \text{ on } D, \quad v = 0 \text{ on } \partial D \]

and \( \phi = a_2 K_1[q \psi] \) and \( K_1 \) denotes the inverse of \(-\mu_1 \Delta - a + 2a_1 u u \). Thus \( \dim(N(K(b) - \mu_1 I)) = 1 \). Suppose that \((\phi, \psi) \in R(K(b) - \mu_1 I) \). Then there exists \( v \) such that

\[ bKv + b_2 K(qv) - \mu_1 v = \psi. \]
Hence

\[ -\mu_1 \Delta v - bv - b_2 q v = -\Delta \psi = -\mu_1^{-1}(b\psi + b_2 q \psi) \quad \text{on } D. \]

Multiplying by \( \psi \) and integrating over \( D \) shows that \( \int_D (b+b_2q)\psi^2 \, dx = 0 \) which is impossible. Hence \( R(K(b) - \mu_1 I) \cap N(K(b) - \mu_1 I) = \{0\} \) and so \( \mu_1 \) is a simple eigenvalue of \( K(b) \). Thus \( i(T(b, \cdot), 0) = -1 \) whenever \( b_0 < b < \lambda_2(-b_2q) \).

Therefore Theorem 3.2 can be applied to \( T \). Thus there exists a continuum \( C_0 \) of solutions of \( (u, v) = T(b, u, v) \) in the \( b - (u, v) \) plane, i.e., in \( \mathbb{R} \times C^1(D) \times C^1(D) \) emanating from \( (b_0, 0, 0) \) and either joining with \( (b, 0, 0) \) where \( I - K(b) \) is not invertible or joining with \( \infty \). Close to the bifurcation point all solutions lie on the curve whose existence we proved by using the Crandall and Rabinowitz theorem. Let \( C_1 \) be the maximal continuum of solutions contained in \( C_0 \) - \{ \( (b(s), s\phi_1 + su(s), s\psi_1 + sv(s)) : -\varepsilon < s < 0 \) \}. Then close to the bifurcation point \( (b_0, 0, 0) \), \( C_1 \) consists of the curve \( \{ (b(s), s\phi_1 + su(s), s\psi_1 + sv(s)) : 0 < s < \varepsilon \} \) and it can be shown by a reflection argument exactly as in Rabinowitz [11] that \( C_1 \) either satisfies one of the same alternatives as \( C_0 \) or contains a pair of points of the form \( (b, u, v) \) and \( (b, - u, - v) \) where \( (u, v) \neq (0, 0) \). Let \( C = \{ (b, u - u, v) : (b, u, v) \in C_1 \} \). Clearly, if \( u, v \geq 0 \) and \( (b, u, v) \in C \), then \( (b, u, v) \) is a solution of system (1.5).

Let \( P_1 = \{ u \in C^1(D) : u(x) > 0 \text{ for } x \in D \text{ and } \partial u / \partial n(x) < 0 \text{ for } x \in \partial D \} \) and let

\[ P = \{ (b, u, v) : b \in \mathbb{R} \text{ and } u, v \in P_1 \}. \]

Clearly \( C \subset P \) in a neighbourhood of the bifurcation point \( (b_0, u_{a}, 0) \). However

**Theorem 4.3.** \( C - \{(b_0, u_{a}, 0)\} \) is not contained in \( P \).

**Proof.** Suppose \( C - \{(b_0, u_{a}, 0)\} \) is contained in \( P \); we shall obtain a contradiction.

By the previous discussion the continuum \( C - \{(b_0, u_{a}, 0)\} \) must

(i) contain points of the form \( (b, u - u, v) \) and \( (b, u + u, - v) \) or

(ii) join up with a bifurcation point of the form \((b, u_{a}, 0)\) where \( b \neq b_0 \) and \( I - K(b) \) is not invertible or

(iii) join \((b_0, u_{a}, 0)\) with \( \infty \).

Since the continuum is contained in \( P \) neither (i) nor (ii) is possible. By Theorem 4.1 and Theorem 4.2 we must have that \( b_0 < b < M \) whenever \( (b, u, v) \in C \). Therefore by Lemma 2.5 there exists a constant \( M_1 > 0 \) such that \( |u(x)|, |v(x)| \leq M_1 \) for all \( x \in D \) whenever \( (b, u, v) \in C \). It follows from standard bootstrapping arguments that \( C \) is bounded in \( \mathbb{R} \times C^1(D) \times C^1(D) \) and so (iii) is also impossible. This is a contradiction and so the continuum is not contained in \( P \).

**Theorem 4.4.** \( C \) joins with \( S_1 \).

**Proof.** Since \( C - \{(b_0, u_{a}, 0)\} \) is not contained in \( P \), there exists \((\hat{b}, \hat{u}, \hat{v}) \in [C - \{(b_0, u_{a}, 0)\}] \cap \partial P \) which is the limit of a sequence \( \{(b_n, u_n, v_n)\} \subseteq C \cap P \). As \( (\hat{b}, \hat{u}, \hat{v}) \in \partial P \), either \( \hat{u} \in \partial P_1 \) or \( \hat{v} \in \partial P_1 \).

Suppose \( \hat{v} \in \partial P_1 \). Then \( \hat{v}(x) \geq 0 \) for \( x \in D \) and either \( \hat{v}(x) = 0 \) for some \( x \in D \) or \( \partial \hat{v} / \partial n(x) = 0 \) for some \( x \in \partial D \). It follows from the second equation in system (1.5) that

\[ -\Delta \hat{v} + [M - b + b_2 \hat{v} - b_2 \hat{u} / (1 + m \hat{u})] \hat{v} = M \hat{v} \geq 0 \quad \text{on } D \]

where \( M \) is a constant chosen sufficiently large so that the term in the square brackets is positive for all \( x \in D \). It follows from the maximum principle that \( \hat{v} \equiv 0 \). A similar but simpler argument shows that if \( \hat{u} \in \partial P_1 \) then \( \hat{u} \equiv 0 \). Thus \( \hat{u} \equiv 0 \) or \( \hat{v} \equiv 0 \).
Suppose that \( \hat{u} = 0 \) and \( \hat{b} = 0 \). Then \((\hat{b}, \hat{u}, \hat{b})\) lies on the branch of trivial solutions \( S_0 = \{(b, 0, 0) : b \in \mathbb{R}\} \). The only nontrivial, nonnegative solutions which are close to \( S_0 \) lie on the semi-trivial branch \( S_1 = \{(b, 0, \nu_b) : b \geq \lambda_1\} \) and so there cannot exist a sequence in \( P \) converging to \((\hat{b}, 0, 0)\). Hence either \( \hat{u} \) or \( \hat{b} \) is nonzero.

Suppose that \( \hat{u} \) is nonzero and \( \hat{b} = 0 \). Then \((\hat{b}, \hat{u}, \hat{b})\) lies on \( S_2 = \{(b, u_a, 0) : b \in \mathbb{R}\} \) and so is a bifurcation point on \( S_2 \) from which bifurcate nontrivial and nonnegative solutions. Therefore by Theorem 3.1 \( \hat{b} \) is such that 1 is an eigenvalue of \( K(\hat{b}) \) with corresponding eigenfunctions which are nonnegative on \( D \). If \((u, v)\) is a nonnegative eigenfunction corresponding to \( \lambda \), then \( v \) satisfies

\[-\Delta v - b_2 q v = \hat{b} v \quad \text{on} \quad D, \quad v = 0 \quad \text{on} \quad \partial D\]

and since \( v \) is nontrivial and nonnegative, it follows that \( \hat{b} = \lambda_1(-b_2q) = b_0 \) and this is impossible.

Thus the only remaining possibility is that \( \hat{b} \) is nonzero and \( \hat{u} = 0 \). In this case \((\hat{b}, \hat{u}, \hat{b})\) must lie on \( S_1 \). Hence \( C \) joins up with \( S_1 \).

It is possible to use our methods to analyze the bifurcation which occurs when \( C \) joins up with \( S_1 \). The arguments involved are very similar to those we develop in the next section and so we omit the details here. In fact \( C \) joins \( S_1 \) when \( b \) is such that \( a = \lambda_1(a_v b) \) (when \( b = b_1 \), say). The argument in Theorem 4.4 shows that, if \((b, u, v) \in C \cap \partial P\), then \((b, u, v) \in S_1 \). Thus \( C \) provides a continuum of nontrivial, nonnegative solutions joining \((b_0, u_a, 0)\) on \( S_2 \) to the point \((b_1, 0, \hat{b})\) on \( S_1 \). In particular we can conclude

**Theorem 4.5.** The system of equations (1.5) has a nontrivial, nonnegative solution provided \( b_0 < b < b_1 \).

5. Structure of solutions with a as bifurcation parameter. We now treat \( a \) as a bifurcation parameter and assume that all the other constants are fixed. The decoupling technique of [2] works in this case. Suppose \( b > \lambda_1 \). Then (1.5) has a continuum of semi-trivial solutions \( S_1 = \{(a, 0, \nu_b) : a \in \mathbb{R}\} \) and it can be proved as in [2] that there is a continuum \( C \) of nontrivial, nonnegative solutions bifurcating from \( S_1 \) at \((\lambda_1(a_v b), 0, \nu_b)\) such that \( C \) does not join up with any other continuum and goes to \( \infty \) as \( a \to \infty \). Thus the following result holds.

**Theorem 5.1.** If \( b > \lambda_1 \), then the system of equations (1.5) has a nontrivial, nonnegative solution provided \( a > \lambda_1(a_v b) \).

The above result could also be established by using an argument similar to that of the preceding section.

Suppose now that \( b < \lambda_1 \). In this case we have the continuum of trivial solutions \( S_0 = \{(a, 0, 0) : a \in \mathbb{R}\} \) and the continuum of semi-trivial solutions \( S_1 = \{(a, u_a, 0) : a \geq \lambda_1\} \). In [2], using decoupling techniques for the classical predator-prey equations it was shown that the stability of the semi-trivial solution \((u_a, 0)\) changes as \( a \) is increased and this indicates that a continuum of nontrivial, nonnegative solutions bifurcates from \( S_1 \). We now use bifurcation techniques similar to those of the preceding section to make a direct investigation of this continuum for the more complicated system (1.5).

As we are linearizing about the same solution as in the previous section, bifurcation seems likely to occur at values of \( a \) such that \( b = \lambda_1(-b_2u_a/(1 + mu_u)) \). Modifying our notation slightly from that used in the previous section in order to highlight the dependence on \( a \), we let \( q_a(x) = u_a(x)/(1 + mu_u(x)) \). Since \( u_a \) is an increasing function of \( a \), \( q_a \) is also an increasing function of \( a \). Clearly \( q_a(x) \leq m^{-1} \) for all \( a \). Hence \( \lambda_1(-b_2q_a) \) is a decreasing function of \( a \) and \( \lambda_1(-b_2q_a) \geq \lambda_1 - b_2/m \) for all \( a \). Thus
\[ \lim_{a \to -\infty} \lambda_1(-b_2q_a) \text{ exists and } \lim_{a \to -\infty} \lambda_1(-b_2q_a) \geq \lambda_1 - b_2/m. \]

By Lemma 2.3 \( u \) converges uniformly to \( u \) on any compact subset of \( D \) and so \( b_2q_a \) converges uniformly to \( b_2/m \). If \( \phi \) denotes the principal eigenfunction of \( -\Delta \) with zero boundary conditions such that \( \int_D \phi^2 \, dx = 1 \), then \( \int_D |\nabla \phi|^2 \, dx = \lambda_1 \). Since \( \lim_{a \to -\infty} \int_D b_2q_a \phi^2 \, dx = b_2/m \int_D \phi^2 \, dx = b_2/m \), it follows that

\[ \lim_{a \to -\infty} \int_D \left( |\nabla \phi|^2 - b_2q_a \phi^2 \right) \, dx = \lambda_1 - b_2/m. \]

Hence by the variational characterisation of eigenvalues \( \lim_{a \to -\infty} \lambda_1(-b_2q_a) \leq \lambda_1 - b_2/m. \)

Thus we have shown

**Lemma 5.1.** \( \lim_{a \to -\infty} \lambda_1(-b_2q_a) = \lambda_1 - b_2/m. \)

Our first theorem shows that if the predator birth rate is too low then no nontrivial solutions exist. This result differs from what occurs in the classical case where, however negative the predator birth rate, nontrivial solutions exist provided the prey birth rate is sufficiently large.

**Theorem 5.2.** If \( (a, u, v) \) is a nonnegative solution of (1.5) with \( v \neq 0 \), then \( b > \lambda_1 - b_2/m. \)

**Proof.** Since

\[ -\Delta v - \left[ b_2 u/(1 + mu) \right] v = bv - b_1 v^2, \]

it follows that

\[ -\Delta v - b_2/m v \leq bv - b_1 v^2. \]

Multiplying by \( v \) and integrating over \( D \), we obtain

\[ -\int_D \Delta v \cdot v \, dx - b_2/m \int_D v^2 \, dx < b \int_D v^2 \, dx. \]

Since \( -\int_D \Delta v \cdot v \, dx \geq \lambda_1 \int_D v^2 \, dx \), it follows that \( b > \lambda_1 - b_2/m. \)

The above theorem shows that, if we fix \( b \leq \lambda_1 - b_2/m \), there can be no bifurcation of nontrivial solutions from \( S_1 \). From now on we suppose that \( b \) is fixed such that \( b > \lambda_1 - b_2/m. \) Since \( a \to \lambda_1(-b_2q_a) \) is a decreasing function which equals \( \lambda_1 \) when \( a = \lambda_1 \) and tends to \( \lambda_1 - b_2/m \) as \( a \to -\infty \), there is a unique value of \( a > \lambda_1 \), say \( a \), such that \( b = \lambda_1(-b_2q_a). \) We show that bifurcation from \( S_1 \) occurs at \( (a, u_a, 0). \)

Motivated by equation (2.7) as in the previous section but now interested in varying \( a \) rather than \( b \), we define \( T : \mathbb{R} \times C^1(D) \times C^1(D) \to C^1(D) \times C^1(D) \) by

\[ T(a, u, v) = (a Ku - 2a_1 K(u_a u) + a_2 K(u q) + KF(a, u, v), b Kv + b_2 K(q q) + KG(a, u, v)) \]

and let \( H = I - T. \) Then \( H(a, u, v) = 0 \) with \( 0 \leq u \leq u_a \) and \( v \geq 0 \) if and only if \( (a, u_a - u, v) \) is a nonnegative solution of (1.5). Clearly \( H(a, 0, 0) = 0 \) and it follows from Lemma 2.2 that \( H \) is a \( C^1 \) function. The Fréchet derivative with respect to \( (u, v) \) is

\[ H_{(u,v)}(a,0,0)(\phi, \psi) = (\phi - a K \phi + 2a_1 K(u \phi) - a_2 K(q \psi), \psi - b K \psi - b_2 K(q \phi)). \]

We have that \( N(H_{(u,v)}(a,0,0)) = \text{span}\{\phi_1, \psi_1\} \) where \( \psi_1 \) is a nonnegative eigenfunction corresponding to the principal eigenvalue \( b = \lambda_1(-b_2q_a) \) of

\[ -\Delta \psi - b_2q_a \psi = b \psi \text{ on } D, \quad \psi_{|\partial D} = 0 \]
and \( \phi_1 = a_2 K_1(q, \psi_1) \) where \( K_1 \) denotes the inverse of \(-\Delta - a + 2a_i u_a\) with Dirichlet boundary conditions. Thus \( \dim N(H_{(u,v)}(\alpha, 0, 0)) = 1 \) and it follows that the codimension of \( R(H_{(u,v)}(\alpha, 0, 0)) = 1 \).

It is easy to check that further differentiation with respect to \( \alpha \) gives

\[
H_{a,(u,v)}(\alpha, 0, 0)(\phi_1, \psi_1) = \left( -K\phi_1 + 2a_1 K(u_a\phi_1) - a_2 K\left[ \psi_1 u_a/(1 + mu)^2 \right], -b_2 K\left[ \psi_1 u_a/(1 + mu)^2 \right] \right)
\]

where \( u_a = du_a/da \). Suppose that \( H_{a,(u,v)}(\alpha, 0, 0)(\phi_1, \psi_1) \in R(H_{(u,v)}(\alpha, 0, 0)) \). Then there exists \( v \) such that

\[
v - bKv - b_2 K(q_dv) = -b_2 K\left[ \psi_1 u_a/(1 + mu)^2 \right]
\]

and so

\[-\Delta v - bv - b_2 q_dv = -b_2 \psi_1 u_a/(1 + mu)^2.\]

Multiplying by \( \psi_1 \) and integrating over \( D \) shows that

\[0 = -b_2 \int_D u_a^2/(1 + mu)^2 \, dx.\]

But by Lemma 2.2 \( u_a^2 \) is positive on \( D \) and so we have a contradiction.

Thus the hypotheses of the Crandall–Rabinowitz theorem are satisfied and so there exists a curve of nontrivial, nonnegative solutions of (1.5) bifurcating from \( S_1 \) at \((\alpha, u\alpha, 0)\). We shall investigate the global nature of this continuum. First we show that bifurcation of nonnegative solutions is to the right.

**Theorem 5.3.** If \((\alpha, u, v)\) is a nontrivial, nonnegative solution of (1.5), then \( \alpha > \alpha \).

**Proof.** Suppose \( \alpha < \alpha \). Then \( b = \lambda_1(q_a) \leq \lambda_1(q_a) \). But by Theorem 4.1 system (1.5) has nontrivial, nonnegative solutions only when \( b > \lambda_1(q_a) \). Thus, if \((\alpha, u, v)\) is nontrivial, \( \alpha > \alpha \).

We now compute \( i(T(\alpha, \cdot), 0) \) so that we can apply our global bifurcation result. This index is \(( -1)^\beta \) where \( \beta \) is the sum of the algebraic multiplicities of the eigenvalues of \( K(\alpha) \) where \( K(\alpha) \) is the compact linear operator

\[K(\alpha)(u, v) = (aKu - 2a_1 K(u_a u) + a_2 K(q_dv), bKv + b_2 K(q_dv)).\]

If \( \mu > 0 \) is an eigenvalue of \( K(\alpha) \), then \( b \) must be an eigenvalue of

\[\mu \Delta v - b_2 q_dv = \lambda v \text{ on } D, \quad v|_{\partial D} = 0.
\]

Conversely, if \( \mu \geq 1 \) and \( b \) is an eigenvalue of (5.1) with corresponding eigenfunction \( v \), then \((u, v)\) is an eigenfunction of \( K(\alpha) \) corresponding to the eigenvalue \( \mu \) where \( u = (-\mu \Delta - a + 2a_i u_a)^{-1} v \), the inverted differential operator corresponding to zero boundary conditions.

Suppose that \( a > a \) and \( \mu \geq 1 \) is an eigenvalue of \( K(\alpha) \). Then \( b \) is an eigenvalue of (5.1) and, since \( \mu \geq 1 \), it follows that \( b > \lambda_1(-b_2 q_a) \). But \( b = \lambda_1(-b_2 q_a) < \lambda_1(-b_2 q_a) \) and this is a contradiction. Hence, if \( a < a \), \( K(\alpha) \) has no eigenvalues \( > 1 \) and so \( i(T(\alpha, \cdot), 0) = 1 \).
Now suppose that \( a \) is such that \( \lambda_1(-b_2q_a) < b < \lambda_2(-b_2q_a) \), i.e. \( a \) lies in an open interval with left-hand end point \( \alpha \). An argument similar to that used in the preceding section shows that \( K(a) \) has a unique eigenvalue \( \mu_1 \) which is greater than 1 and that this eigenvalue is simple. Therefore \( i(T(a, \cdot), 0) = -1 \) and so Theorem 3.2 can be applied to \( T \).

By arguments similar to those used in the preceding section it can be proved that there exists a continuum \( C \) in \( \mathbb{R} \times C^1(D)\times C^1(D) \) emanating from \( (a, u_a, 0) \) such that

(i) if \( (a, u, v) \in C \), then \( (u - u, v) = T(a, u - u, v) \);
(ii) if \( (a, u, v) \in C \) and \( u, v \geq 0 \), then \( (a, u, v) \) is a solution of (1.5);
(iii) close to the bifurcation point \( (a, u_a, 0) \), \( C \) consists of the points \( (a, u, v) \) on the curve given by the Crandall and Rabinowitz theorem with \( v \geq 0 \).

We now show that \( C \) does not join up with any other continuum but extends to \( a = \infty \).

**Theorem 5.4.** (i) If \( (a, u, v) \in C - \{(a, u_a, 0)\} \), then \( u, v \in P_1 \), i.e. \( u, v > 0 \) on \( D \) and \( \partial u /\partial n, \partial v /\partial n < 0 \) on \( \partial D \).

(ii) \( \{a : (a, u, v) \in C \} = [a, \infty) \).

**Proof.** (i) Suppose that \( C \) contains a point \( (a, u, v) \neq (a, u_a, 0) \) which lies outside of \( P \). Then there exists a point \( (\hat{a}, \hat{u}, \hat{v}) \in C - \{(a, u_a, 0)\} \cap \partial P \) which is the limit of a sequence of points \( \{(a_n, u_n, v_n)\} \) in \( C \cap \partial P \). It follows as in the previous section that \( \hat{u} \equiv 0 \) or \( \hat{b} \equiv 0 \).

Suppose that \( \hat{u} \equiv 0 \) and \( \hat{b} \equiv 0 \). Then \((\hat{a}, \hat{u}, \hat{v}) = (a, 0, 0) \) and so \((\hat{a}, \hat{u}, \hat{v}) \) lies on the trivial branch of solutions \( S_0 \). The only nontrivial, nonnegative solutions which are close to \( S_0 \) lie on the semi-trivial branch \( S_1 = \{(a, u_a, 0) : a \geq \lambda_1 \} \) and so there cannot exist a sequence in \( C \cap P \) converging to \((\hat{a}, \hat{u}, \hat{v}) \). Therefore it is impossible that both \( \hat{u} \) and \( \hat{b} \) are identically zero.

Suppose that \( \hat{u} \equiv 0 \). Then

\[- \Delta \hat{b} = b\hat{b} - b_1 \hat{b}^2 \quad \text{on} \quad D, \quad \hat{b}|_{\partial D} = 0\]

and so, since \( b < \lambda_1 \), \( \hat{b} \equiv 0 \). Therefore \( \hat{u} \) is not identically zero.

Suppose that \( \hat{b} \equiv 0 \). Then \((\hat{a}, \hat{u}, \hat{b}) \in S_1 \) and they bifurcate from \((a, u, b) \) nontrivial, nonnegative solutions. Therefore by Theorem 3.1 \( \hat{a} \) is such that 1 is an eigenvalue of \( K(\hat{a}) \) with corresponding eigenfunctions which are nonnegative on \( D \). Thus \( \hat{a} \) must be such that \( b = \lambda_1(-b_2q_a) \) when \( a = \hat{a} \). Hence \( \hat{a} = a \) and \((\hat{a}, \hat{u}, \hat{b}) = (a, u_a, 0) \) which is impossible.

Therefore, if \( (a, u, v) \in C - \{(a, u_a, 0)\} \), then \((a, u, v) \in P \).

(ii) \( C \) must satisfy one of the three alternatives discussed in the preceding section. Because of (i) above, \( C \) contains no pairs of points of the form \((a, u_a - u, v) \) and \((a, u_a + u, -v) \) and \( C \) cannot join up with another bifurcation point of the form \((a, u_a, 0) \) on \( S_1 \). Hence \( C \) joins \((a, u_a, 0) \) to \( \infty \). By Theorem 5.3 we have that \( a \geq a \) whenever \((a, u, v) \in C \). Lemmas 2.3 and 2.5 show that there exist a constant \( M_1(a) \) such that, if \( (a, u, v) \in C \), then \( |u(x)|, |v(x)| \leq M_1(a) \) for all \( x \in D \). Bootstrapping arguments imply that there exists a constant \( M(a) \) such that \( \|u\|, \|v\| \leq M(a) \) where \( \| \| \) denotes the norm in \( C^1(D) \). Hence the only way for \( C \) to approach \( \infty \) in \( \mathbb{R} \times C^1(D)\times C^1(D) \) is by \( a \) becoming unbounded. Since \( \{a : (a, u, v) \in C \} \) is connected, it must equal \([a, \infty) \).

Thus we obtain the following theorem on the existence of solutions of (1.5) to complement Theorem 5.2.

**Theorem 5.5.** Suppose \( b > \lambda_1 - b_2/m \). Then (1.5) has a nontrivial, nonnegative solution if and only if \( a > \alpha \) where \( b = \lambda_1(-b_2q_a) \) i.e. if and only if \( a \) is sufficiently large so that \( b > \lambda_1(-b_2q_a) \).
REFERENCES


