Existence and uniqueness of positive radial solutions for the Lane–Emden system

Robert Dalmasso*

Laboratoire LMC-IMAG, Equipe EDP, Tour IRMA, BP 53, F-38041 Grenoble Cedex 9, France

Received 22 April 2003; accepted 24 February 2004

Abstract

In this article, we consider (component-wise) positive radial solutions of a weakly coupled system of elliptic equations in a ball with homogeneous nonlinearities. The existence is well-known in general: We give a result for the remaining cases. The uniqueness is less studied: We complement the known results.

Keywords: Semilinear elliptic system; Positive radial solution

1. Introduction

In this paper, we study the existence and the uniqueness of (component-wise) positive radial solutions of the semilinear elliptic system with homogeneous Dirichlet data

\[
\begin{align*}
\Delta u + v^q &= 0 \quad \text{in } B_R, \\
\Delta v + u^p &= 0 \quad \text{in } B_R, \\
u = v &= 0 \quad \text{on } \partial B_R,
\end{align*}
\]

(1.1)

where \( B_R \) denotes the open ball of radius \( R \) centered at the origin in \( \mathbb{R}^n (n \geq 1) \) and \( p, q > 0 \). Eq. (1.1) is a natural extension of the well-known Lane–Emden equation and thus referred to as the Lane–Emden system.

When \( q = 1 \) and \( p \in (0, 1) \cup (1, +\infty) \) the uniqueness of a positive radial solution of problem (1.1) was established in [2] (see also the references therein). When \( p, q > 0 \) and \( pq < 1 \) the uniqueness of a positive radial solution of problem (1.1) is just

* Tel.: +33-4-76-63-57-38; fax: +33-4-76-63-12-63.
E-mail address: robert.dalmasso@imag.fr (R. Dalmasso).
a particular case of a result obtained in [5]. Finally, when \( n = 1, \ q \geq 1 \) and \( p > 1 \), the uniqueness of a positive solution follows from a general result given in [3]. Therefore, it only remains to study the case where \( p, q \in (0,1) \cup (1, +\infty) \) with \( pq > 1 \) and the case where \( p, q > 0 \) with \( pq = 1 \).

We have the following theorem.

**Theorem 1.1.** (i) Let \( p, q > 0 \) with \( pq \neq 1 \). Then the Lane–Emden system (1.1) has at most one positive radial solution.

(ii) Let \( p, q > 0 \) with \( pq = 1 \). Assume that the Lane–Emden system (1.1) has a positive radial solution \((u,v)\). Then all positive radial solutions are given by \((\theta^qu,\theta v)\), where \( \theta > 0 \) is an arbitrary constant.

When \( p, q \geq 1, \ pq > 1 \) and
\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \quad \text{if } n \geq 3,
\]
the existence of a positive solution was established in [1,6,8] for more general nonlinearities (see also [3] when \( n = 1 \)). Moreover, it is well-known (see [7,11]) that when \( n \geq 3, \ p, q > 0 \) and
\[
\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n},
\]
problem (1.1) has no positive solution. If \( p, q > 0 \) and \( pq < 1 \), the existence of a positive solution is a particular case of a result proved in [5]. Therefore, it remains to examine the existence of positive radial solutions of problem (1.1) in the cases described in the following theorem.

**Theorem 1.2.** (i) If \( p > 1, \ q \in (0,1) \) satisfy (1.2) and \( pq > 1 \), then the Lane–Emden system (1.1) has a positive radial solution.

(ii) Let \( p, q > 0 \) with \( pq = 1 \). Then there exists \( R > 0 \) such that the Lane–Emden system (1.1) has a positive radial solution.

2. **Proof of Theorem 1.1.** (i) Let \((u,v)\) and \((w,z)\) be two positive radial solutions of problem (1.1). Let \( s \) and \( t \) be defined by
\[
s = 2 \frac{q + 1}{pq - 1} \quad \text{and} \quad t = 2 \frac{p + 1}{pq - 1}.
\]
For \( \lambda > 0 \) we set
\[
\tilde{w}(r) = \lambda^sw(\lambda r) \quad \text{and} \quad \tilde{z}(r) = \lambda^t z(\lambda r), \quad 0 \leq r \leq R/\lambda.
\]
We have
\[
\begin{cases}
\Delta \tilde{w} + \tilde{z} = 0, & 0 \leq r \leq R/\lambda, \\
\Delta \tilde{z} + \tilde{w} = 0, & 0 \leq r \leq R/\lambda,
\end{cases}
\]
and
\[
\tilde{w}(R/\lambda) = \tilde{z}(R/\lambda) = 0.
\]
where $\Delta$ denotes the polar form of the Laplacian

$$\Delta = r^{1-n} \frac{d}{dr} \left( r^{n-1} \frac{d}{dr} \right).$$

Choose $\lambda$ such that $\lambda \hat{w}(0) = u(0)$. Then we have

$$\hat{w}(0) = u(0). \tag{2.2}$$

We want to show that

$$\hat{z}(0) = v(0). \tag{2.3}$$

Suppose that $\hat{z}(0) < v(0)$. If there exists $a \in (0, \min(R, R/\lambda))$ such that $\hat{z} - v < 0$ on $[0, a)$ and $(\hat{z} - v)(a) = 0$, then $\hat{w} - u > 0$ on $(0, a]$. Indeed assume the contrary. Then there exists $b \in (0, a]$ such that $(\hat{w} - u)(b) \leq 0$. Since $\Delta(\hat{w} - u) = vq - zq > 0$ on $[0, b)$, the maximum principle implies that $\hat{w} - u < 0$ on $[0, b)$, a contradiction with (2.2).

Now we have $\Delta(\hat{z} - v) = uq - wq < 0$ on $(0, a]$ and the maximum principle implies that $\hat{z} - v > (\hat{z} - v)(a) = 0$ on $[0, a)$, a contradiction. Thus $\hat{z} - v < 0$ on $[0, \min(R, R/\lambda)]$.

Since

$$\hat{z} - v = \begin{cases}
-v(R/\lambda) & \text{if } \lambda > 1, \\
0 & \text{if } \lambda = 1, \\
\hat{z}(R) & \text{if } \lambda < 1,
\end{cases} \tag{2.4}$$

we deduce that $\lambda > 1$. As before we show that $\hat{w} - u > 0$ on $(0, \min(R, R/\lambda)]$. We have

$$\hat{w} - u = \begin{cases}
-u(R/\lambda) & \text{if } \lambda > 1, \\
0 & \text{if } \lambda = 1, \\
\hat{w}(R) & \text{if } \lambda < 1,
\end{cases}$$

hence $\lambda < 1$ and we have a contradiction. The case $\hat{z}(0) > v(0)$ can be handled in the same way. Thus (2.3) is proved.

Now we define the functions $U$, $W$, $F$ and $G_n$ by

$$U(r) = (u(r), v(r)), \quad 0 \leq r \leq R,$$

$$W(r) = (\hat{w}(r), \hat{z}(r)), \quad 0 \leq r \leq R/\lambda,$$

$$F(x, y) = (y^q, x^p), x, y \geq 0$$

and

$$G_n(r, s) = \begin{cases}
 r - s & \text{if } n = 1, \\
 s \ln \left( \frac{r}{s} \right) & \text{if } n = 2, \\
 \frac{s}{n-2} \left( 1 - \left( \frac{s}{r} \right)^{n-2} \right) & \text{if } n \geq 3, 
\end{cases} \tag{2.4}$$

for $0 \leq s \leq r$. Using (2.2) and (2.3) and the fact that $u'(0) = w'(0) = v'(0) = \hat{z}'(0) = 0$ we easily obtain

$$U(r) - W(r) = \int_0^r G_n(r, s)(F(W(s)) - F(U(s))) \, ds.$$
for $r \in [0, \min(R, R/\lambda)]$. When $p, q \geq 1$, $F$ is locally Lipschitz continuous, and using Gronwall’s lemma we obtain $U = W$ on $[0, \min(R, R/\lambda)]$. When $p$ or $q \in (0, 1)$, let $a \in (0, \min(R, R/\lambda))$ be fixed. Then $u(0) \geq u(r) \geq u(a) > 0$, $\tilde{w}(0) = u(0) \geq \tilde{w}(r) > \tilde{w}(a) > 0$, $v(0) \geq v(r) \geq v(a) > 0$ and $\tilde{z}(0) = v(0) \geq \tilde{z}(r) \geq \tilde{z}(a) > 0$ for $r \in [0, a]$. Since $F$ is locally Lipschitz continuous on $(0, +\infty) \times (0, +\infty)$, as before we obtain $U = W$ on $[0, a]$. By continuity we get $U = W$ on $[0, \min(R, R/\lambda)]$. Now we deduce that $\lambda = 1$ and thus $(u; v) = (w, z)$ on $[0, R]$. 

(ii) Let $(u, v)$ be a positive radial solution of problem (1.1). Then, for any $\theta > 0$, $(w, z) = (\theta^q u, \theta v)$ is a positive radial solution of problem (1.1). Now let $(w, z)$ be a positive radial solution of (1.1). Choose $\theta > 0$ such that $\theta^q u(0) = w(0)$ and define $\tilde{w} = \theta^q u$, $\tilde{z} = \theta v$. Then $(\tilde{w}, \tilde{z})$ is a positive radial solution of (1.1) such that $\tilde{w}(0) = w(0)$. Arguing as in part (i) we show that $\tilde{z}(0) = z(0)$ and that $(\tilde{w}, \tilde{z}) = (w, z)$.

**Remark 2.1.** Notice that our technique also applies when there is a homogeneous dependence on the radius $|x|$. More precisely, for $p, q > 0$ and $pq \neq 1$, the following system:

\[
\begin{align*}
    \Delta u + |x|^s v^\theta &= 0 \quad \text{in } B_R, \\
    \Delta v + |x|^t u^p &= 0 \quad \text{in } B_R, \\
    u = v = 0 \quad \text{on } \partial B_R,
\end{align*}
\]  

where $\mu, \nu \geq 0$, has at most one positive radial solution $(u, v)$. Indeed, the arguments are the same with $s$ and $t$ in (2.1) replaced by

\[
    s = \frac{2(q + 1) + v + q \mu}{pq - 1} \quad \text{and} \quad t = \frac{2(p + 1) + \mu + p v}{pq - 1}.
\]

Now let $p, q > 0$ with $pq = 1$. Assume that problem (2.5) has a positive radial solution $(u, v)$. Then all positive radial solutions are given by $(\theta^q u, \theta v)$, where $\theta > 0$ is an arbitrary constant.

3. **Proof of Theorem 1.2.** We shall use a two-dimensional shooting argument for the ordinary differential system associated to radial solutions of (1.1) [3,4,9,10]. We only assume for the moment that $p, q > 0$. Since we are only interested in positive solutions we may extend $v \to v^\theta$ and $u \to u^p$ to $\mathbb{R}$ by setting

\[
g(v) = \begin{cases} v^\theta & \text{if } v > 0, \\
0 & \text{if } v \leq 0,
\end{cases} \quad \text{and} \quad f(u) = \begin{cases} u^p & \text{if } u > 0, \\
0 & \text{if } u \leq 0.
\end{cases}
\]

We introduce the one-dimensional (singular if $n \geq 2$) initial value problem

\[
\begin{align*}
    \Delta u(r) + g(v(r)) &= 0, \quad r > 0, \\
    \Delta v(r) + f(u(r)) &= 0, \quad r > 0, \\
    u(0) = \alpha, v(0) = \beta, u'(0) = v'(0) = 0,
\end{align*}
\]  

where $\alpha > 0$, $\beta > 0$. 

We shall need a series of lemmas. We begin with a local existence and uniqueness result. The proof is standard (see [3] if \( n = 1 \) and [9,10] when \( n \geq 3 \)).

**Lemma 3.1.** For any \( \alpha > 0, \beta > 0 \) there exists \( R > 0 \) such that problem (3.1) on \([0,R]\) has a unique solution \((u,v) \in (C^2[0,R])^2\).

In view of Lemma 3.1, for any \( \alpha, \beta \) problem (3.1) has a unique local solution: Let \([0,R_{x,\beta}]\) denote the maximum interval of existence of that solution \((R_{x,\beta} = +\infty \text{ possibly})\). If \( 0 < q < 1 \) or \( 0 < p < 1 \) the uniqueness of the solution could fail at any point \( r \) where \( v(r) = 0 \) or \( u(r) = 0 \). In this case \( R_{x,\beta} \) could also depend on the particular solution itself. Define

\[
P_{x,\beta} = \{ s \in (0,R_{x,\beta}); u(x,\beta, r)v(x,\beta, r) > 0 \quad \forall r \in [0,s] \},
\]

where \((u(x,\beta, \cdot), v(x,\beta, \cdot))\) is a solution of (3.1) in \([0,R_{x,\beta}]\). Set

\[
r_{x,\beta} = \sup P_{x,\beta}.
\]

Notice that the solution is unique on \([0,r_{x,\beta}]\), so \( r_{x,\beta} \) depends only on \( \alpha, \beta \).

The next lemma can be proved by direct integration of system (3.1).

**Lemma 3.2.** We have \( u'(x,\beta, r) < 0 \) and \( v'(x,\beta, r) < 0 \) for \( r \in (0,R_{x,\beta}) \).

**Lemma 3.3.** For any \( \alpha, \beta > 0 \) we have \( R_{x,\beta} > r_{x,\beta} \).

**Proof.** If not, there exist \( \alpha > 0 \) and \( \beta > 0 \) such that \( r_{x,\beta} = R_{x,\beta} \). Suppose first that \( R_{x,\beta} < \infty \). Noting \( u = u(x,\beta, \cdot) \) and \( v = v(x,\beta, \cdot) \) we have

\[
0 \leq u \leq \alpha \quad \text{and} \quad 0 \leq v \leq \beta \quad \text{on} \quad [0,R_{x,\beta}].
\]

Since

\[
u'(r) = -r^{1-n} \int_0^r s^{n-1} v(s)^p \, ds \quad \text{and} \quad u'(r) = -r^{1-n} \int_0^r s^{n-1} u(s)^q \, ds
\]

for \( r \in (0,R_{x,\beta}) \), we conclude that \( u, v, u' \) and \( v' \) are bounded on \([0,R_{x,\beta}]\) and we get a contradiction with the definition of \( R_{x,\beta} \). Now assume that \( R_{x,\beta} = +\infty \). When \( n \geq 3 \) the result follows from Corollary 1.2 in [9]. When \( n=1 \) we have \( u'' < 0 \) on \([0, +\infty)\).

We deduce that

\[
u'(r) \leq u'(1) < 0 \quad \text{for all} \quad r \geq 1,
\]

from which we get

\[
u(r) \leq u(1) + u'(1)(r - 1) \quad \text{for all} \quad r \geq 1.
\]

Thus, we can find \( r \geq 1 \) such that \( u(r) < 0 \) and we obtain a contradiction. Now if \( n=2 \), we have \((ru'(r))' < 0 \) on \((0, +\infty)\). We deduce that

\[
u'(r) \leq u'(1) < 0, \quad \text{for all} \quad r \geq 1,
\]

from which we get

\[
u(r) \leq u(1) + u'(1) \ln r, \quad \text{for all} \quad r \geq 1.
\]

Thus we can find \( r \geq 1 \) such that \( u(r) < 0 \) and we obtain a contradiction.
Proposition 3.1. For any \( x > 0 \) there exists a unique \( \beta > 0 \) such that \( u(x, \beta, r_{x, \beta}) = v(x, \beta, r_{x, \beta}) = 0 \).

Proof. We first prove the uniqueness. Let \( x > 0 \) be fixed. Suppose that there exist \( \beta > \gamma > 0 \) such that \( u(x, \beta, r_{x, \beta}) = v(x, \beta, r_{x, \beta}) = u(x, \gamma, r_{x, \gamma}) = v(x, \gamma, r_{x, \gamma}) = 0 \). Using the same arguments as in the proof of (2.3) we obtain a contradiction.

Now we prove the existence. Suppose that there exists \( x > 0 \) such that for any \( \beta > 0 \) \( u(x, \beta, r_{x, \beta}) \neq 0 \) or \( v(x, \beta, r_{x, \beta}) \neq 0 \). Define the sets

\[
B = \{ \beta > 0; u(x, \beta, r_{x, \beta}) = 0 \text{ and } v(x, \beta, r_{x, \beta}) > 0 \}
\]

and

\[
C = \{ \beta > 0; u(x, \beta, r_{x, \beta}) > 0 \text{ and } v(x, \beta, r_{x, \beta}) = 0 \}.
\]

The proof of the proposition is completed by using the next two lemmas which contradict the fact that \( (0, +\infty) = B \cup C \). \( \square \)

Lemma 3.4. (i) Suppose \( B \neq \emptyset \). Then there exists \( m > 0 \) such that \( m \leq \inf B \).

(ii) Suppose \( C \neq \emptyset \). Then there exists \( M > 0 \) such that \( M \geq \sup C \).

Lemma 3.5. \( B \) and \( C \) are open.

Proof of Lemma 3.4. We have

\[
u(x, \beta, r) = x - \int_0^r G_n(r, s)g(v(x, \beta, s))\,ds, \quad 0 \leq r < R_{x, \beta}, \tag{3.2}
\]

and

\[
v(x, \beta, r) = - \int_0^r G_n(r, s)f(u(x, \beta, s))\,ds, \quad 0 \leq r < R_{x, \beta}, \tag{3.3}
\]

where \( G_n \) is defined in (2.4).

(i) Let \( \beta \in B \). Lemma 3.2 and (3.2) imply

\[
r_{x, \beta} \geq \left( \frac{2nx}{\beta^q} \right)^{1/2}, \tag{3.4}
\]

and from (3.3) we get

\[
\beta \geq \int_0^{r_{x, \beta}} G_n(r_{x, \beta}, s)u(x, \beta, s)^p\,ds. \tag{3.5}
\]

Suppose that \( \inf B = 0 \) and let \( (\beta_j) \) be a sequence in \( B \) decreasing to zero. Then \( r_{x, \beta_j} \to +\infty \) by (3.4). From (3.5) we deduce that

\[
\beta_j \geq \int_0^1 G_n(r_{x, \beta_j}, s)u(x, \beta_j, s)^p\,ds \tag{3.6}
\]

for \( j \) large. Using Lemma 3.2 and (3.2) we have

\[
u(x, \beta_j, r) \geq x - \frac{\beta_j^q}{2n} \geq \frac{x}{2} \tag{3.7}
\]

for \( r \in [0, 1] \) and \( j \) large. From (3.6) and (3.7), we get \( \beta_j \geq c \) for \( j \) large where \( c > 0 \) is independent of \( j \). This gives a contradiction.
(ii) Suppose that \( \sup C = +\infty \) and let \((\beta_j)\) be a sequence in \( C \) increasing to \(+\infty\). By virtue of Lemma 3.2 we have

\[
0 < u(z, \beta_j, r) \leq z \quad \text{for} \quad r \in [0, r_{x, \beta_j}].
\]

(3.8) and (3.8) imply that \( r_{x, \beta_j} \to +\infty \) as \( j \to +\infty \). Then we can assume that \( r_{x, \beta_j} \geq 1 \) for all \( j \) and that

\[
z^p \leq \beta_j \quad \text{for all} \quad j.
\]

(3.9) Lemma 3.2, (3.3), (3.8) and (3.9) imply

\[
\frac{2n - 1}{2n} \beta_j \leq v(z, \beta_j, r) \leq \beta_j \quad \text{for} \quad r \in [0, 1]
\]

and using (3.2) we deduce that \( u(z, \beta_j, 1) \leq z - (2n - 1)q/\beta_j^q/(2n)^{q+1} \). But then \( u(z, \beta_j, 1) < 0 \) for \( j \) large contradicting (3.8).

**Remark 3.1.** When \( n = 1 \) and \( p, q \geq 1 \) Lemma 3.5 is proved in [3]. It is easily seen that the proof also holds when \( n \geq 2 \). Now if \( 0 < q < 1 \) and \( p > 1 \), the arguments given in [3] do not apply since \( g \) is no longer Lipschitz continuous at \( 0 \); thus one loses continuous dependence on the initial data at \( r = r_{x, \beta} \), a fact which was essential in the proof. We shall use a result [10, Lemma 2.4] which applies in the Lipschitz and non-Lipschitz cases. In fact \( n \geq 3 \) in [10], but the result used also holds when \( n = 1 \) or 2. Finally notice that \( f \) and \( g \) in [10] (see also [9]) are

\[
f(x) = |x|^p \quad \text{and} \quad g(x) = |x|^q
\]

for \( x \in \mathbb{R} \), but the proof is the same with our definition of \( f \) and \( g \).

**Proof of Lemma 3.5.** As explained in Remark 3.1 we shall use a result established in [10]. Define

\[
X = \{(\gamma, \beta) \in (0, +\infty) \times (0, +\infty); v(\gamma, \beta, r, \gamma, \beta) > u(\gamma, \beta, r, \gamma, \beta) = 0\}
\]

and

\[
Y = \{(\gamma, \beta) \in (0, +\infty) \times (0, +\infty); u(\gamma, \beta, r, \gamma, \beta) > v(\gamma, \beta, r, \gamma, \beta) = 0\}.
\]

\( X \) and \( Y \) are open (see [9,10]). Now let \( h : \mathbb{R}^2 \to \mathbb{R} \) denote the second projection on \( \mathbb{R} \). Since \( B = h(\{x\} \times (0, +\infty) \cap X) \) and \( C = h(\{x\} \times (0, +\infty) \cap Y) \), we deduce that \( B \) and \( C \) are open and the lemma is proved.

Now we can complete the proof of Theorem 1.2.

(i) Let \( z > 0 \) be fixed. By Proposition 3.1 there exists a unique \( \beta > 0 \) such that \( u(z, \beta, r_{x, \beta}) = v(z, \beta, r_{x, \beta}) = 0 \). With \( s \) and \( t \) defined in (2.1) we set

\[
w(r) = \left(\frac{r_{x, \beta}}{R}\right)^s u \left( z, \beta, \frac{r_{x, \beta}}{R} r \right) \quad \text{and} \quad z(r) = \left(\frac{r_{x, \beta}}{R}\right)^t v \left( z, \beta, \frac{r_{x, \beta}}{R} r \right), \quad 0 \leq r \leq R.
\]

Then \((w,z)\) is a positive radial solution of the Lane–Emden system (1.1).

(ii) follows from Proposition 3.1.

**Remark 3.2.** Notice that, by Lemma 3.3, \( X \) and \( Y \) are precisely \( A \) and \( B \) in [10] (\( B \) and \( A \) in [9]).
References