A Rigorous Numerical Method in Infinite Dimensions

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Sarah Day

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A Rigorous Numerical Method in Infinite Dimensions

Approved by:

Professor Konstantin Mischaikow, Advisor

Professor Luca Dieci

Professor Peter Mucha

Professor Andrzej Swiech

Professor Margaret Symington

Professor Greg Turk
(College of Computing)

Date Approved

For my mother,

in loving memory

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SUMMARY

Dynamical systems may exhibit many beautiful and highly complicated behaviors which are often difficult to capture analytically. With recent advances in computing power, numerical analysis is a useful approach, either as an initial investigation or to study systems for which direct analysis is difficult or even impossible. However, numerical computations require a number of properties which some of the more interesting systems do not initially possess. In the very least, they require finite dimensional systems on discretized, compact domains. In addition, sensitive dependence, one of the defining properties of chaotic systems, leads errors to blow up in time. This makes straight forward simulations of the system problematic if not misleading. The subject of this work is to present a general method for reducing a (possibly infinite dimensional) dynamical system to one that is more computationally friendly yet still captures the essential features of the original system. This discussion is based on joint work with Konstantin Mischaikow and Oliver Junge in [3].

Two key observations give hope that a reduced system which captures the interesting dynamical properties exists. The first observation is that invariant sets often contain functions which are more regular than the typical functions of the natural phase space. This regularity, which has been shown to be a property of a large number of systems, allows for a restriction of the studied domain to a compact subset (see [11] and earlier references therein). The second observation is that dynamical objects are often low dimensional (e.g. fixed points, periodic orbits, homoclinic and heteroclinic orbits, horseshoes). An appropriate Galerkin projection of the full system onto a finite dimensional space should capture such low dimensional objects. This projection is a restriction to what are sometimes refered to as "finite determining modes". Originally discussed by Foias and Prodi in [9], finite determining modes have been shown to exist, at least abstractly, for a wide variety of systems (again, see [11] and earlier references therein). As previously noted, these restrictions are

essential if one wants to study a system numerically.

The final ingredient in this approach is a topological tool, the Conley index. This index, a generalization of Morse theory, is able to tolerate bounded errors in proving the existence of dynamical objects of various stability types. Performing the Galerkin projection and discretizing the studied domain, not to mention truncations introduced by the computer itself, all contribute to errors in the numerical study of the system. *Error is unavoidable and should be considered as an intrinsic element of the studied system.* For this reason, we actually study multivalued systems which associate to each element of the domain an image which is a set of elements. This multivalued system reflects the optimal knowledge of the projection of the full system, given that bounded errors are present. The Conley index is an algebraic topological invariant which can be computed for the multivalued system. This index can detect the existence of the "coarse" dynamics which exists for any single-valued system contained within the multivalued system. Finally, by verifying a few extra conditions, the index information may be lifted to the full, original system.

This approach has been used to study both continuous and discrete systems. It was first applied in the context of dissipative PDEs by Mischaikow and Zgliczyński in [34] to study the Kuramoto-Sivashinsky equation. More recently, we have used this approach to prove the existence of periodic orbits, connecting orbits, and chaotic symbolic dynamics for the Kot-Schaffer map, an infinite dimensional, discrete system from ecology. For clarity, most of the following discussion will also refer to the Kot-Schaffer map. In addition, Chapter 4 serves as an illustration, in the finite dimensional setting of the Hénon map, of some of the key computational techniques. It is important to note, however, that the properties one needs to use this approach are satisfied by a wide class of systems. In particular, given an appropriate orthonormal basis (which in the high dimensional setting serves as a starting point for the Galerkin projection), the regularity assumption previously discussed, and that any nonlinearities are polynomial in nature, one may directly apply the following procedure. Each of these properties will be discussed in greater detail in the following chapters.