Numerics and the Conley Index: GAIO, CHomP, and two examples

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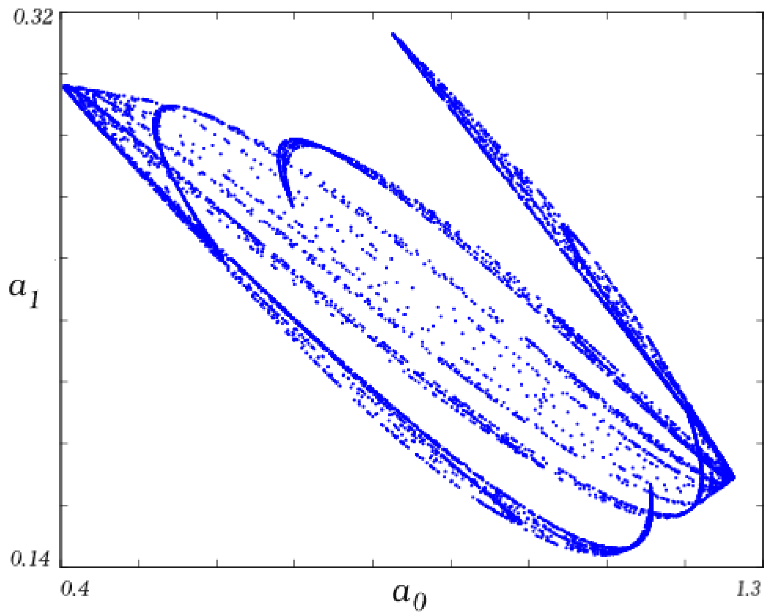
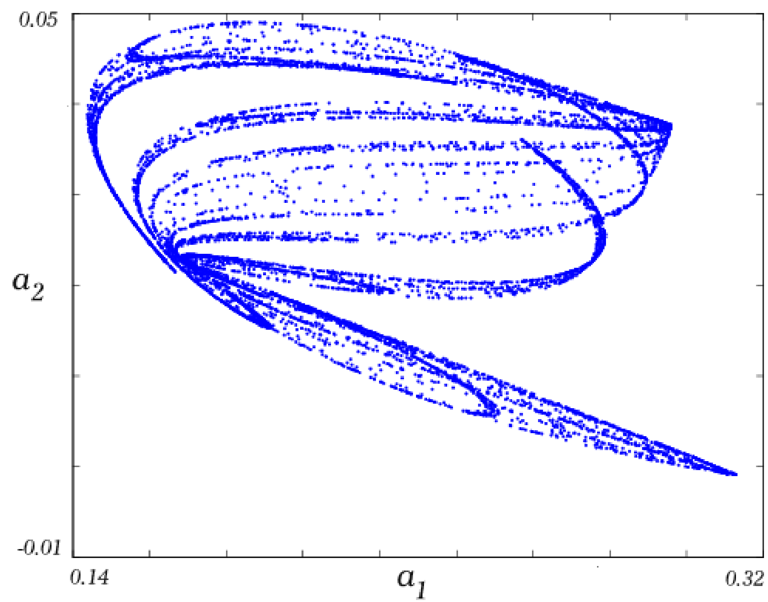
http://www.math.gatech.edu/~sday/thesis.pdf

(and references therein)

Computational considerations





1. high (or infinite) dimensional, continuous spaces do not fit easily into a computer
2. sensitive dependence causes errors to blow-up in time
3. interpretation

Numerics and the Conley index

Three properties make the Conley index particularly well-suited for numerical studies:

1. may tolerate a priori bounded error

2. computable in low dimensions (GAIO, CHomP)

3. provides rigorous information about the existence and structure of invariant sets

Goal: *Develop computational techniques based on the Conley index to rigorously prove results about the structure of specific nonlinear dynamical systems.*

The Method

1. Reduce the system to one which is computationally - friendly. *(Galerkin projection, regularity, discretization)*

2. Extract rigorous information from numerical computations on finite dimensional systems. *(GAIO, CHomP, Conley Index)*

3. (For high dimensional systems) lift the results of the finite dimensional computations to the full, original system. *(Conley Index, continuity, compactness)*

balance computational costs with numerical precision

Two examples

1. Hénon map
   - simplified model of (chaotic) weather patterns
   - 2-dimensional
   - periodic orbits, connecting orbits, chaotic symbolic dynamics
   - an example where simulation is misleading

2. Kot-Schaffer model
   - models populations with discrete growth/dispersal phases
   - infinite-dimensional
   - chaotic symbolic dynamics

Computation: GAIO and CHomP

Two software packages for treating discrete dynamical systems using cubical structures:

- GAIO
  - set-oriented numerical investigation of dynamical systems
  - attractors and invariant manifolds, invariant measures, ...
  - http://math-www.uni-paderborn.de/~agdellnitz/gaio/
- CHomP
  - algorithmic homology computations
  - cubical structures
  - http://www.math.gatech.edu/~chom/

Discretizing space (GAIO)

Partition phase space $W = \prod_{k=0}^{m-1}[x_k^-, x_k^+]$ into a finite number of boxes by iteratively bisecting W with respect to the j-th coordinate direction, $j = 0, 1, \ldots$.

The number of boxes grows exponentially in the dimension m of W.

Instead, keep only a box covering of the maximal invariant set (numerical effort essentially determined by dimension of the maximal invariant set).

Discretizing space (GAIO)

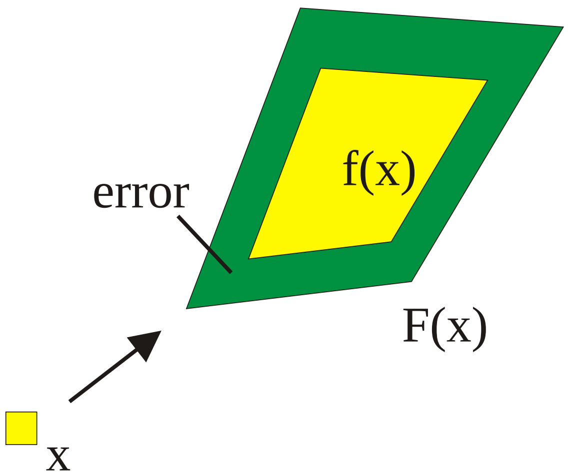
The boxes may be efficiently stored in a binary tree

- a coordinate direction is assigned to each level of the tree
- the root (depth = 0) corresponds to the box W
- all nodes at a depth k correspond to a subset $\mathcal{B}_k$ of a cubical grid on W

Finite representation of the map (GAIO)

In a natural way, the (multivalued) map F on W defines a multivalued map $\mathcal{F}$ on $\mathcal{B}_k$: For $B \in \mathcal{B}_k$, let

$$\mathcal{F}(B) = \{\tilde{B} \in \mathcal{B}_k | F(B) \cap \tilde{B} \neq \emptyset\}$$



"linear term + Lipschitz estimate"

The image may also be enlarged to incorporate additional error bounds.

$\mathcal{F}$ is stored as a (sparse) matrix or a graph, called the transition matrix, $M_{\mathcal{F}}$, or transition graph, $\mathcal{G}_{\mathcal{F}}$.

Find interesting structures in $\mathcal{M}_{\mathcal{F}}$ (or $\mathcal{G}_{\mathcal{F}}$) using graph theoretic techniques:

- k-periodic points of $\mathcal{F}$ are identified by nonzero diagonal entries of $M_{\mathcal{F}}^k$
- more generally, recurrent sets of $\mathcal{F}$ are given by strongly connected components of $\mathcal{G}_{\mathcal{F}}$
- in this graph, connecting orbits of $\mathcal{F}$ can be identified by shortest path algorithms (e.g. Dijkstra's algorithm);
- compute the maximal invariant set of $\mathcal{F}$ (Szymzcak).

"Growing" isolating neighborhoods

Let $\tilde{\mathcal{S}}$ be a guess for an invariant set of $\mathcal{F}$.

Algorithm (Growing Isolating Neighborhood)

$\mathcal{S}$ = `make_isolated(`$\tilde{\mathcal{S}}$`)`

$\mathcal{S}$ `:=` $\mathsf{Inv}(\tilde{\mathcal{S}}, \mathcal{F})$

`while` $o(\mathcal{S}) \not\subset \tilde{\mathcal{S}}$

$\tilde{\mathcal{S}}$ `:=` $\tilde{\mathcal{S}} \cup o(\mathcal{S})$

$\mathcal{S}$ `:=` $\mathsf{Inv}(\tilde{\mathcal{S}}, \mathcal{F})$

`if` $\mathcal{S} \subset \mathsf{int}\,|o(\mathcal{S})|$ `return` $\mathcal{S}$

`else return` $\emptyset$

($o(\mathcal{S})$ denotes the one box neighborhood of $\mathcal{S}$)

This algorithm returns a combinatorial isolated invariant set $\mathcal{S}$ for $\mathcal{F}$ with combinatorial isolating neighborhood $\mathcal{I} := o(\mathcal{S})$. In the case where $|\mathcal{S}|$ touches the boundary of W, the empty set is returned.

Constructing index pairs (Szymczak)

Given an isolated invariant set $\mathcal{S}$, let

$$\mathcal{N}_1 := \mathcal{S} \cup \mathcal{F}(\mathcal{S})$$

and

$$\mathcal{N}_0 := \mathcal{N}_1 \setminus \mathcal{S}$$

Then, $|\mathcal{N}| = (|\mathcal{N}_1|, |\mathcal{N}_0|)$ is an index pair for any continuous selector f of F (and of $|\mathcal{F}|$).

Modification

There are two considerations which prompt us to modify this construction.

1. The index pairs constructed in this way may be too large (especially for sets with high dimensional, strongly unstable behavior).

2. Computation of the index map on this pair could introduce "folding effects". Computation of the index map involves computing an auxiliary (or image) pair which contains the image of the original index pair. The inclusion of the index pair into the image pair must induce an isomorphism in relative homology for the computed index map to be well-defined. This fails when the *image* of a box in $\mathcal{N}_0$ folds back to intersect to $o(\mathcal{S}) \setminus \mathcal{N}_1$.

Modified index pairs

The following algorithm produces smaller index pairs (restricted to a one-box neighborhood of $\mathcal{S}$ as motivated by Szymczak's definition of a weak index pair) which also prohibit the wrapping effect.

Algorithm (Modified Combinatorial Index Pair)

$[\mathcal{P}_1, \mathcal{P}_0]$ `= build_ip(`$\mathcal{S}$`)`
    $\mathcal{P}_0$ `:=` $\emptyset$
    $\mathcal{E}$ `:=` $(\mathcal{F}(\mathcal{S}) \cap o(\mathcal{S})) \setminus \mathcal{S}$
    while $\mathcal{E} \neq \emptyset$
        $\mathcal{P}_0$ `:=` $\mathcal{P}_0 \cup \mathcal{E}$
        $\mathcal{E}$ `:=` $(\mathcal{F}(\mathcal{P}_0) \cap o(\mathcal{S})) \setminus \mathcal{P}_0$
    $\mathcal{P}_1$ `:=` $\mathcal{S} \cup \mathcal{P}_0$
    return $[\mathcal{P}_1, \mathcal{P}_0]$

Weak index pairs

Recall: A pair $N = (N_1, N_0)$ of compact sets is a weak index pair for $f : Y \to Y$ if and only if $\mathrm{Inv}(I, F) \subset \mathrm{int}(I)$ where $I := \mathrm{cl}(N_1 \setminus N_0)$ and the index map $f_N : (N_1/N_0, [N_0]) \to (N_1/N_0, [N_0])$ given by

$$f_N([x]) = \begin{cases} [f(x)] & \text{if } x,\ f(x) \in N_1 \setminus N_0 \\ [N_0] & \text{otherwise} \end{cases} \tag{1}$$

is a continuous map.

This weaker version is sufficient for defining the index.

Auxiliary pair

Define the auxiliary pair $[\mathcal{Q}_1, \mathcal{Q}_0]$ by

$$\mathcal{Q}_1 := \mathcal{F}(\mathcal{P}_1) \text{ and } \mathcal{Q}_0 := \mathcal{Q}_1 \setminus \mathcal{S}$$

Note that by construction, $[\mathcal{P}_1, \mathcal{P}_0] = [\mathcal{Q}_1 \cap o(\mathcal{S}), \mathcal{Q}_0 \cap o(\mathcal{S})]$. Equivalently,

$$(|\mathcal{P}_1|, |\mathcal{P}_0|) = (|\mathcal{Q}_1| - A, |\mathcal{Q}_0| - A)$$

where $A = |\mathcal{Q}_1 - o(\mathcal{S})| \subset |\mathcal{Q}_0|$. By excision, the inclusion map $i : (|\mathcal{P}_1|, |\mathcal{P}_0|) \hookrightarrow (|\mathcal{Q}_1|, |\mathcal{Q}_0|)$ induces an isomorphism i_* in relative homology. This property ensures that we prevent the folding effect and, therefore, may compute a well-defined index map.

Computing the index, CHomP

By construction, $\mathcal{F} : [\mathcal{P}_1, \mathcal{P}_0] \to [\mathcal{Q}_1, \mathcal{Q}_0]$ respects this pair structure, i.e. $\mathcal{F}(\mathcal{P}_1) \subset \mathcal{Q}_1$ and $\mathcal{F}(\mathcal{P}_0) \subset \mathcal{Q}_0$.

For $i = 0, 1$, let F_i be $|\mathcal{F}|$ restricted to $|\mathcal{P}_i|$.

Let Γ_{F_i}, be the graph of $F_i : |\mathcal{P}_i| \to |\mathcal{Q}_i|$. By construction of $\mathcal{F}$, the natural projection $p : (\Gamma_{F_1}, \Gamma_{F_0}) \to (|\mathcal{P}_1|, |\mathcal{P}_0|)$ induces an isomorphism in homology.

Then the map $f_* : H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \to H_*(|\mathcal{P}_1|, |\mathcal{P}_0|)$ induced by any continuous selector f of F in homology is $f_* = q_* \circ (p_*)^{-1}$ where q is the natural projection $(\Gamma_{F_1}, \Gamma_{F_0}) \to (|\mathcal{Q}_1|, |\mathcal{Q}_0|)$.

Efficient algorithms for cubical graph reductions which preserve homology, in addition to other computational techniques, are also used during this procedure.

Finally, if the inclusion map $i : (|\mathcal{P}_1|, |\mathcal{P}_0|) \hookrightarrow (|\mathcal{Q}_1|, |\mathcal{Q}_0|)$ induces an isomorphism in homology then the index map is

$$f_{P*} = (i_*)^{-1} \circ f_*$$

This additional property that the inclusion map induces an isomorphism may be verified independently when working with a general combinatorial index pair $[\mathcal{P}_1, \mathcal{P}_0]$. Alternatively, this property is ensured by the construction of the modified combinatorial index pair.

Using the index

The Lefschetz number on the pointed space $(P_1/P_0, [P_0])$ is

$$\Lambda(I, f) := \sum_{n=0}^{\infty} (-1)^n tr f_{P,n}$$

where $I = P_1 \setminus P_0$.

Theorem. If $\Lambda(I, f) \neq 0$, then f has a fixed point in I.

Proof. (Szymzcak) The point $[P_0]$ is a strong sink/isolated fixed point with index 1. The fixed point index of the set of all fixed points is $\Lambda(I, f) + 1$. If this total index is not equal to the index of $[P_0]$, then there is a fixed point in I (different from $[P_0]$).

Similarly,

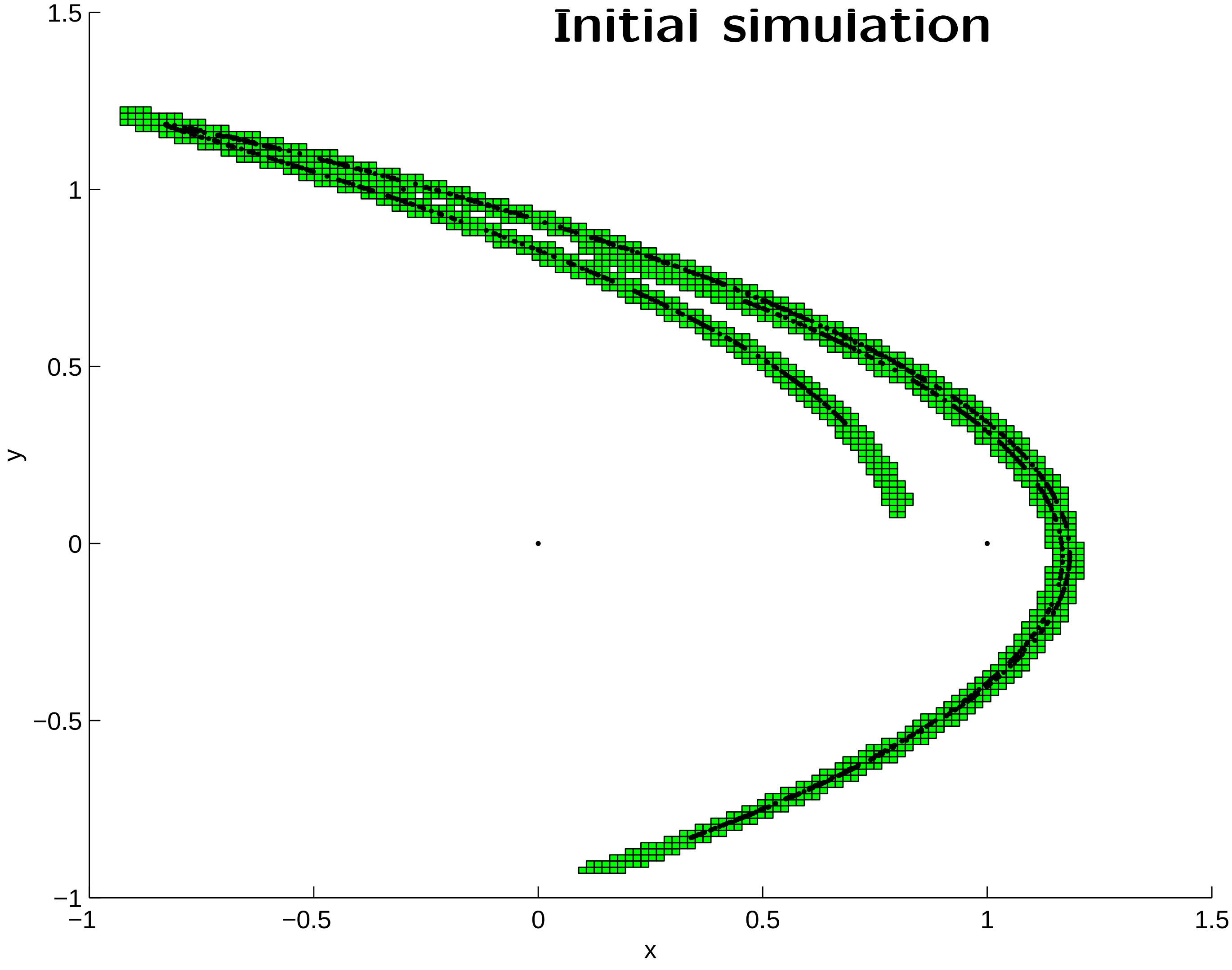
1. if $\Lambda(I, f^k) \neq 0$, then f has a periodic orbit of period k
2. if $\Lambda(I, f_{C_n} \circ \ldots \circ f_{C_1}) \neq 0$, then f has a periodic orbit traveling through components $C_1, \ldots, C_n$

Example 1: the Hénon map

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + y/5 \\ 5bx \end{pmatrix}$$

With parameter values $a = 1.3$ and $b = 0.2$.


Initial simulation
1.5
1
0.5
0
−0.5
−1
y
−1
−0.5
0
0.5
1
1.5
x

Finite representation

From these simulations, we choose a box to initialize a tree in GAIO.

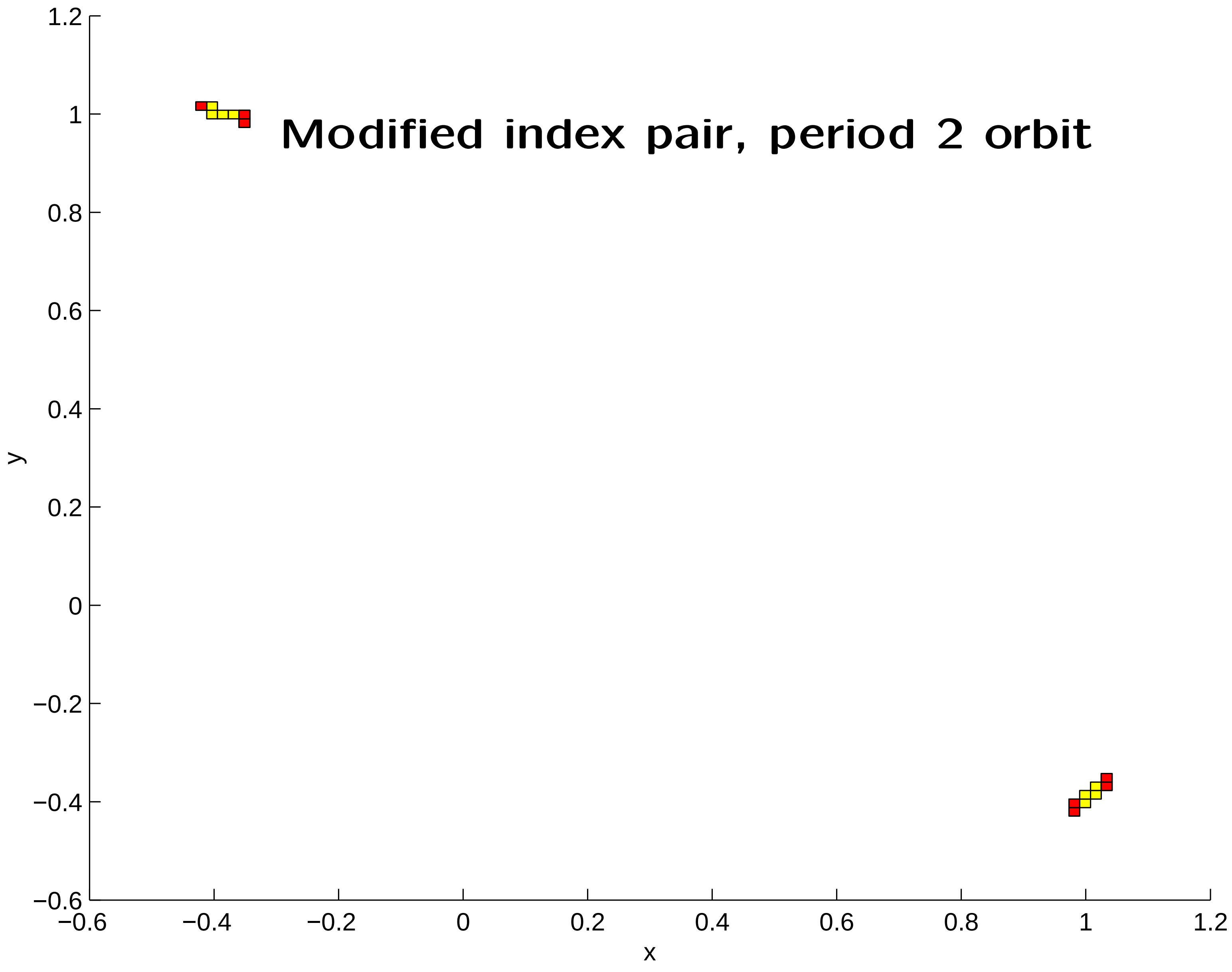
For our computations we choose to initially subdivide each box 14 times (subdividing 7 times in each of the two directions). This results in 1188 boxes, each of box radius $[0.0087, 0.0087]$ covering the maximal invariant set and a transition matrix, $M_{\mathcal{H}}$, on these boxes.

Period 2 orbit

- Initial guess: nonzero entries of the diagonal of $M_{\mathcal{H}}^2$ (boxes 553 and 78)

- "grow" this initial two box collection into a combinatorial isolated invariant set $\mathcal{S}$ for $\mathcal{H}$

- construct the corresponding modified combinatorial index pair, $[\mathcal{P}_1, \mathcal{P}_2]$



Modified index pair, period 2 orbit
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
y
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
x

Period 2 orbit, index

Compute the index

- construct auxiliary pair $[\mathcal{Q}_1, \mathcal{Q}_2]$
- use CHomP, with $[\mathcal{P}_1, \mathcal{P}_2]$, $[\mathcal{Q}_1, \mathcal{Q}_2]$, and $M_{\mathcal{H}}$ on $\mathcal{P}_1$

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^2,\ 0,\ 0,\ \ldots).$$

For an appropriate choice of basis,

$$h_{P,1} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Each generator of $H_1(|\mathcal{P}_1|, |\mathcal{P}_0|)$ lies in a distinct connected component of $|\mathcal{P}_1| \setminus |\mathcal{P}_0|$.

Period 2 orbit, proof

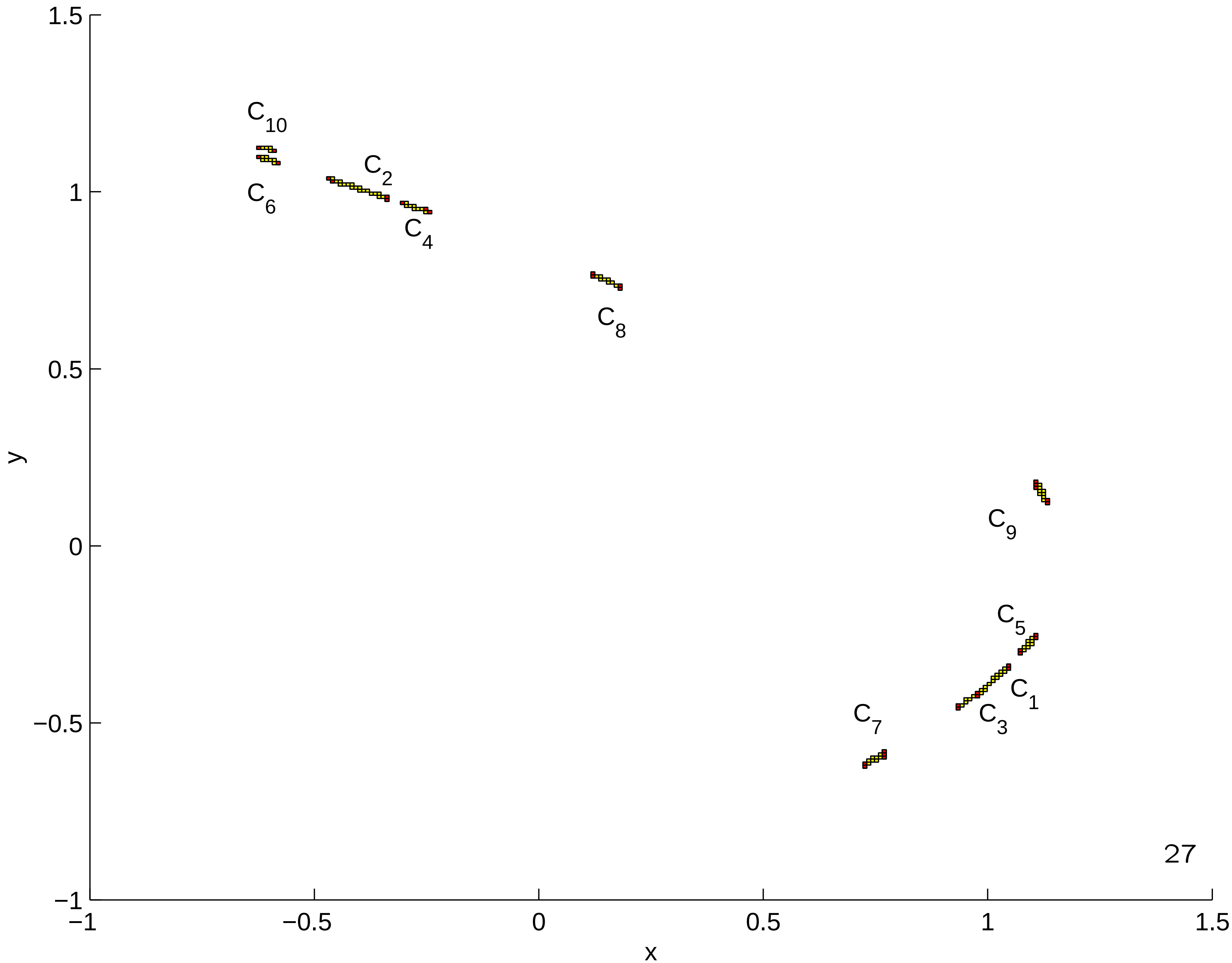
Theorem. There exists a periodic orbit of minimal period two for the Hénon map with elements in each of the distinct connected components of $\mathcal{S}$.

Proof. The Lefschetz number $\Lambda(\mathcal{I}, h^2) = -tr(h^2_{P,1}) = -2$ is nonzero so $\mathcal{S}$ contains a periodic point of period two. Furthermore, this orbit has minimal period two since the transition graph on the two connected components of $\mathcal{S}$ prohibits there being a fixed point.

A connecting orbit

Initial guess: an orbit connecting a period 2 orbit to a period 4 orbit in $M_{\mathcal{H}}$ at depth 16

Grow isolating neighborhood, construct modified index pair.


1.5
1
0.5
0
−0.5
−1
y
−1
−0.5
0
0.5
1
1.5
x
C10
C6
C2
C4
C8
C9
C5
C1
C3
C7

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{10},\ 0,\ 0,\ \ldots).$$

$$h_{P,1} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Each of the 10 generators lies in a distinct component.

Connecting orbit, proof

Theorem. There exists an orbit for the Hénon map which limits in forwards time to a box neighborhood of a period 4 orbit, and in backwards time to a box neighborhood containing a period 2 orbit.

Proof. The periodic orbits exist:

$$\operatorname{tr} h^2_{C_1 \cup C_2, 1} = -2 \text{ and } \operatorname{tr} h^4_{C_7 \cup \ldots \cup C_{10}, 1} = -4.$$

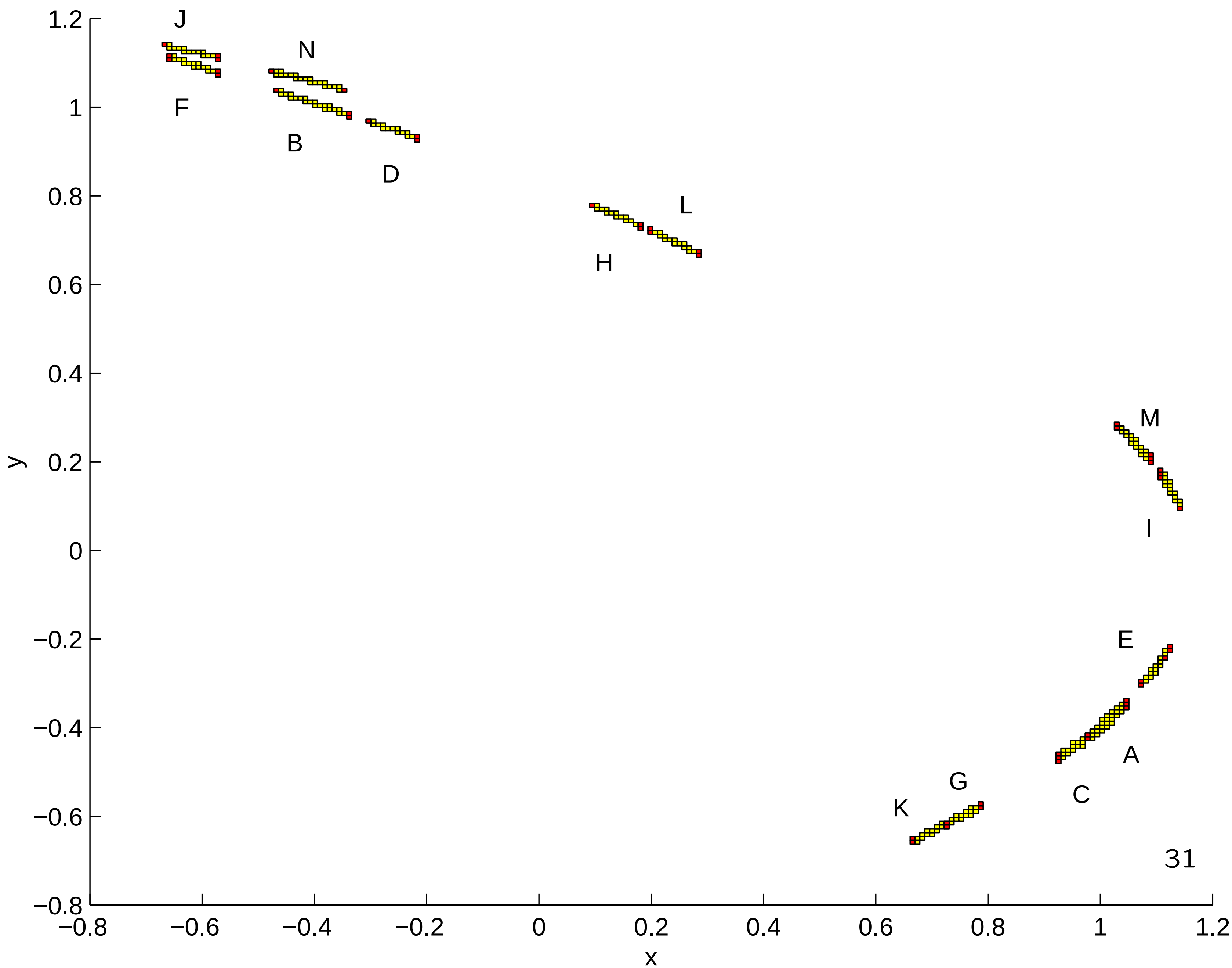
h_{P*} is not shift equivalent to $h_{C_1 \cup C_2 *} \oplus h_{\cup_{i \in \{7,8,9,10\}} C_i *}$, so the Conley indices of $\mathrm{Inv}(|\mathcal{S}|, h)$ and $\mathrm{Inv}(\cup_{i \in \{1,2,7,8,9,10\}} C_i, h)$ are different. Hence $\mathrm{Inv}(|\mathcal{S}|, h) \neq \mathrm{Inv}(\cup_{i \in \{1,2,7,8,9,10\}} C_i, h)$

This, in addition to the transition information given by $M_{\mathcal{H}}$, completes the proof.

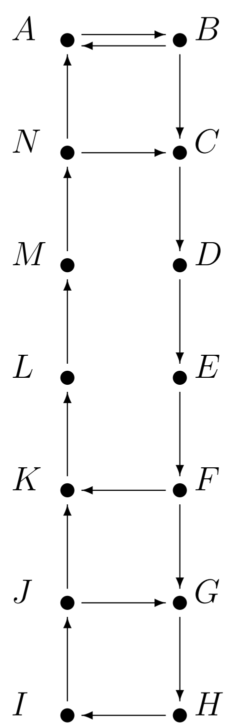
Chaotic symbolic dynamics, 1d-unstable

Initial guess: a period four orbit, a period two orbit, and two connecting orbits in the transition graph given by $M_{\mathcal{H}}$ at depth 16

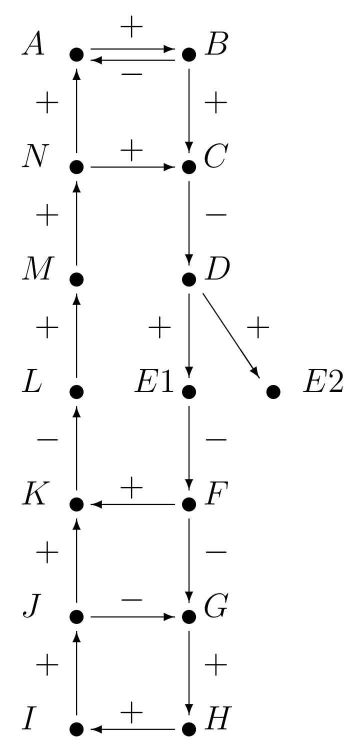
Grow isolating neighborhood, construct modified index pair.


J
N
F
B
D
L
H
M
I
E
A
C
G
K
31
x
y
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{15},\ 0,\ 0,\ \ldots).$$



transition graph on components



index map on generators

Theorem There is a set contained in $|\mathcal{S}|$, on which h is semi-conjugate to the symbol subshift given by the transition graph.

Proof.

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ \rho \downarrow & & \downarrow \rho \\ \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \end{array}$$

$\Sigma_T = \{(Z_i)_i | (Z_i, Z_{i-1})\text{an edge in } T\}$

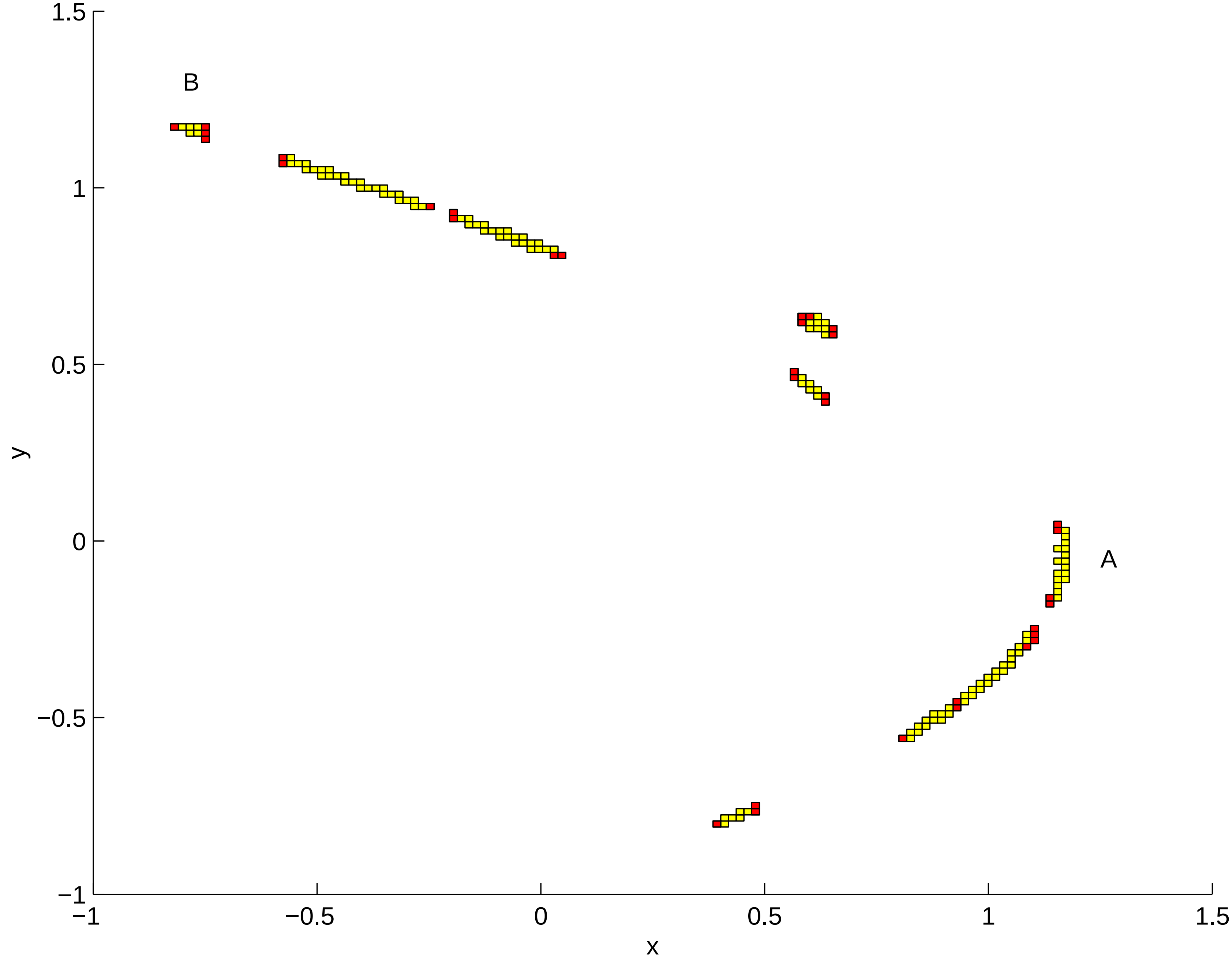
$\rho(x) = \{Z_i | f^i(x) \in Z_i\}$

σ is left shift by one symbol

- Study e.g. $\Lambda(I, h^{42}_{JIHG} h_F \ldots h_D \ldots h_K)$. Since for each periodic symbol sequence, the corresponding Lefschetz number is nonzero, there exists at least one corresponding periodic orbit in S.
- ρ is continuous and S is compact. Therefore, ρ maps onto Σ_T, the closure of periodic orbits.

Transition graph vs the index

Initial guess: connecting orbit from a period two orbit to a fixed point given by $M_{\mathcal{H}}$ at depth 14



1.5
1
0.5
0
−0.5
−1
y
−1
−0.5
0
0.5
1
1.5
x
B
A

The computed index is

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^9,\ 0,\ 0,\ \ldots)$$

$$h_{P,1} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & {\color{red}0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

$h_{P,1}$ folds the only generator in Component A (column 5) and maps it trivially to the generator in Component B (row 6).

Subdividing $\mathcal{S}$ at depth 14 to a depth of 24, $M_{\mathcal{H}}$ prohibits a connecting orbit of this type.

In infinite dimensions

Outline:

1. Reduce the system to a finite-dimensional system for computation *(Galerkin projection, regularity, discretization)*

2. Extract rigorous information from numerical computations on the finite-dimensional system. *(GAIO, CHomP, Conley Index)*

3. Lift the results of the finite dimensional computations to the full, original system. *(lifting set: Conley Index, continuity, compactness)*

Example 2: an infinite dimensional map

The Kot-Schaffer growth-dispersal model for plants:

$$\begin{aligned} \Phi : L^2((-\pi,\pi),\mathbb{R}) &\to L^2((-\pi,\pi),\mathbb{R}) \\ \psi &\mapsto \Phi(\psi) \end{aligned}$$

$$\Phi(\psi)(x) = \int_{-\pi}^{\pi} D(x,y) g[\psi](y) dy,$$

growth operator: $g[\psi]= \mu\psi\,(1 - c\psi)$, where c^{-1} gives the local carrying capacity, and $\mu > 0$ is the (birth - death) rate.

dispersal kernel: $D(x,y)= \Sigma_k b_k \exp(ik|x-y|)$, e.g. $b_k = 2^{-k}$

Fourier expansion

The Fourier expansion of Φ yields the maps:

$$a_k \mapsto \mu b_k \left[a_k + \sum_{l,m} c_l a_m a_{k-l-m} \right]$$

where $\{c_n\}$ and $\{a_n\}$ are the expansions of c and a respectively.

In general, for a monomial ca^p,

$$\sum_{n_0,\ldots,n_{p-1} \in \mathbf{Z}} c_{n_0} a_{n_1} \ldots a_{n_{p-1}} a_{k-(n_0+\ldots+n_{p-1})}$$

Regularity

For the Kot-Schaffer system,

$$\begin{aligned} |<\Phi[a],\phi_k>| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} b_k\phi_k(x)g[a](x)dxdy\Big| \\ &= |b_k|\Big|\int_{-\pi}^{\pi}\phi_k(x)g[a](x)dx\Big| \\ &\le |b_k|\|\phi_k\|\|g[a]\| \\ &\le C_{g,a}|b_k| \end{aligned}$$

In particular, if $|b_k| \le \frac{B}{b^k}$ for some constants $B > 0$, $b > 1$, then for sufficiently large k,

$$|<\Phi[a],\phi_k>| \le \frac{A}{b^k}$$

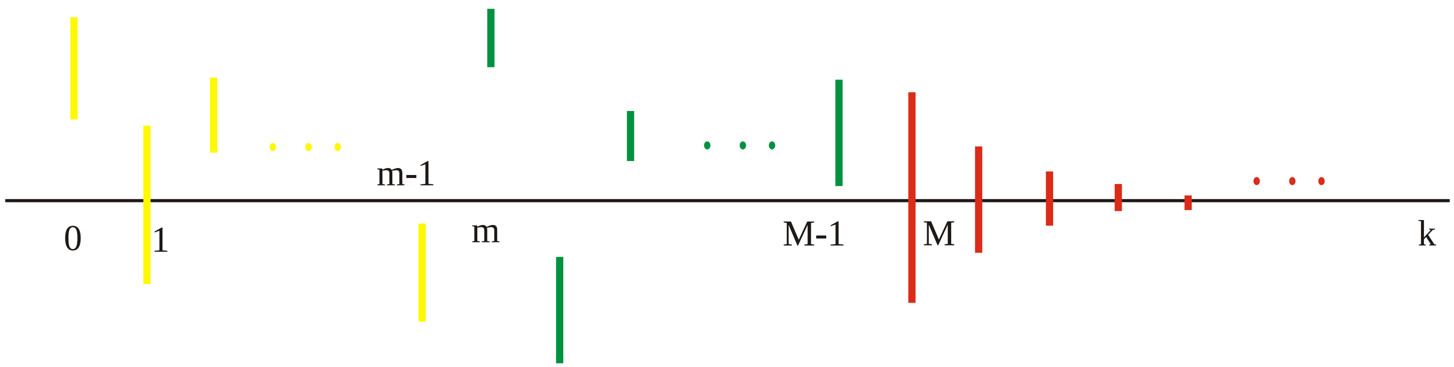
where $A = C_{g,a}B$.

Restricted domain

We will study the maps on subsets of the form

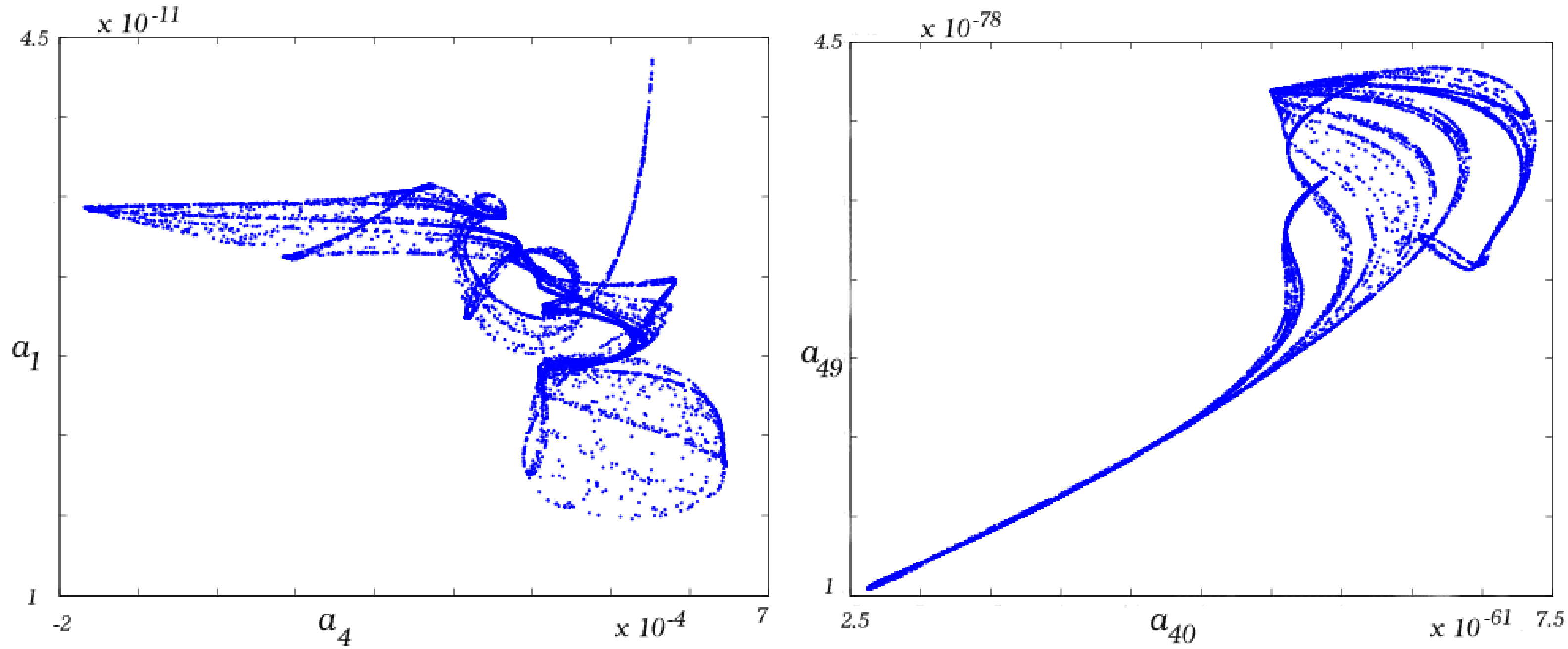
$$Z = \prod_k [a_k^-, a_k^+]$$

where for some constants $m, M > 0$,



variables explicit bounds asymptotic bounds $\frac{A_s}{s^k}$

Choosing an appropriate M



Projections onto modes 5 and 11 (order 10^{-4}), and modes 50 and 51 (order 10^{-61}). $M = 50$, $[a_k^-, a_k^+] = \left[\frac{-1}{2^k}, \frac{1}{2^k}\right]$ for $k \geq M$

Error

$$a'_k = f_k^{(m)}(a_0, \ldots, a_{m-1}) + f_k^{(m+)}(a)$$

Suppose exponential decay, $|a_k| \le \frac{A}{s^k}$

Lemma 1. If $c_n \in \tilde{c}_n := \frac{C}{s^n}[-1, 1]$, then

$$\sum_{n_0, n_1, \ldots, n_{p-1} \in \mathbf{Z}} \tilde{c}_{n_0} \tilde{a}_{n_1} \ldots \tilde{a}_{n_{p-1}} \tilde{a}_{k-(n_0+\ldots+n_{p-1})} \subseteq \frac{\alpha^p A^p C}{s^k} \left(\frac{b}{\beta}\right)^k [-1, 1]$$

where $1 < \frac{b}{\beta} < s$ and $\alpha \ge \frac{2}{\ln s} + \frac{b}{\beta \ln \frac{b}{\beta}}$

(e.g. $\frac{b}{\beta} = \frac{2s}{s+1}$)

Choosing an appropriate m

m	3	4	5	6
$\epsilon_k^{(m+)}$	0.0226 0.0022 0.0022	0.0192 0.0018 0.0005 0.0090	0.0059 0.0009 0.0002 0.0006 0.0023	0.0015 0.0002 0.0001 0.0003 0.0001 0.0006

Errors induced by neglecting the higher order modes for different projection dimensions. $m = 6$

Definition. A set $Z = N \times V$ is a *lifting set* if

1. $N = \prod_{j=0}^{\infty}[x_j^-, x_j^+]$ is an isolating neighborhood for the finite-dimensional, mulitivalued system given by the map $P_m\Phi(\cdot, V)$

2. $V = \prod_{j=m}^{\infty}[x_j^-, x_j^+]$ and for each $j \geq m$, either

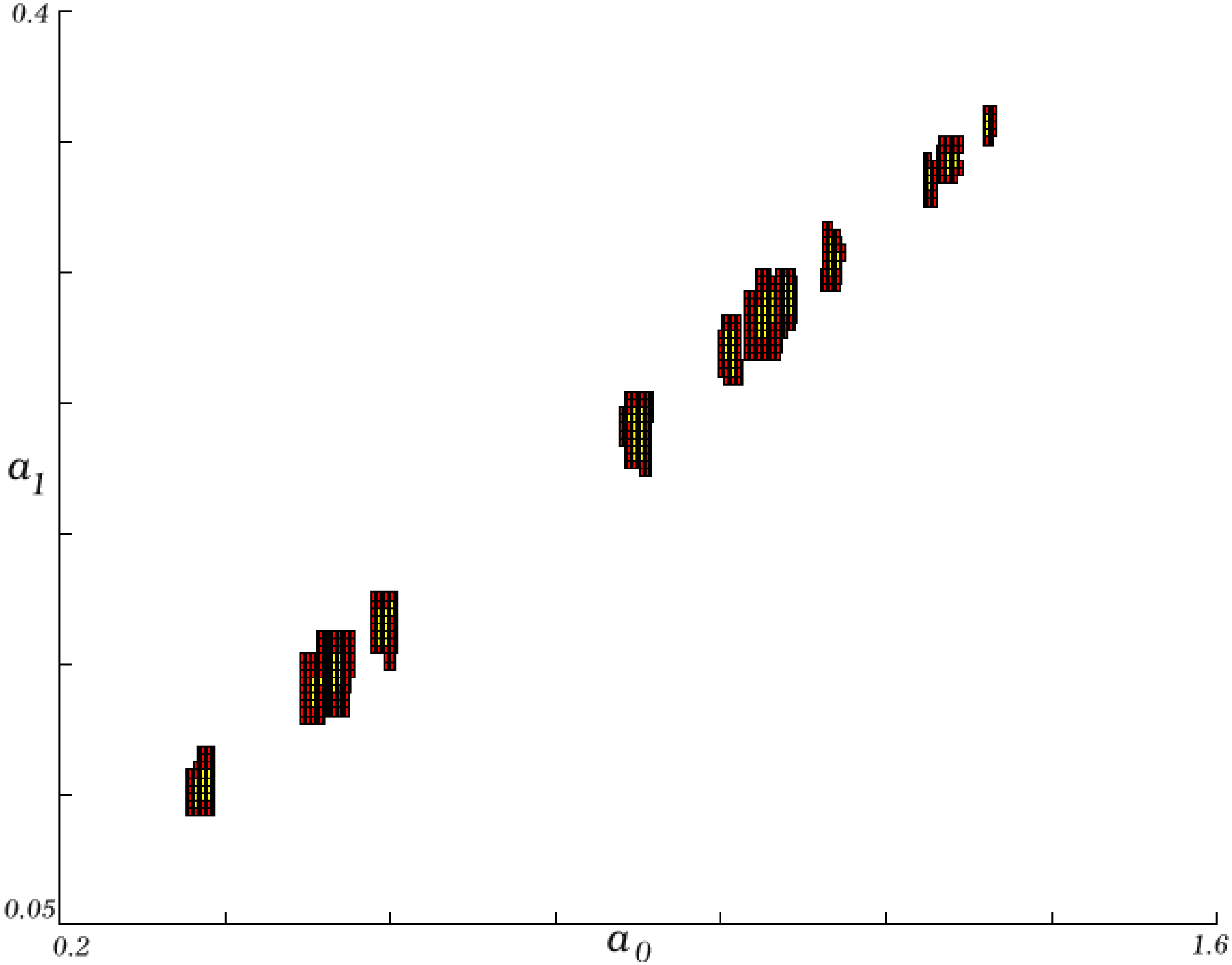
   (a) (contraction) $\pi_j\Phi(Z) \subset (x_j^-, x_j^+)$, or

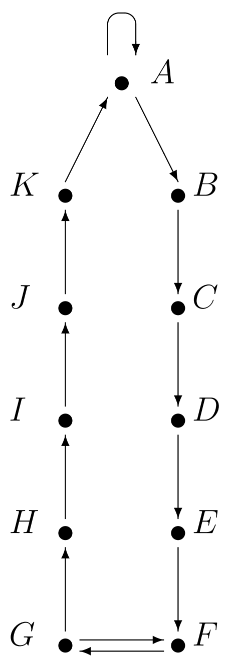
   (b) (expansion) $\pi_j\Phi(x) \notin [x_j^-, x_j^+]$ for all $x \in Z$ with $x_j = x_j^{\pm}$

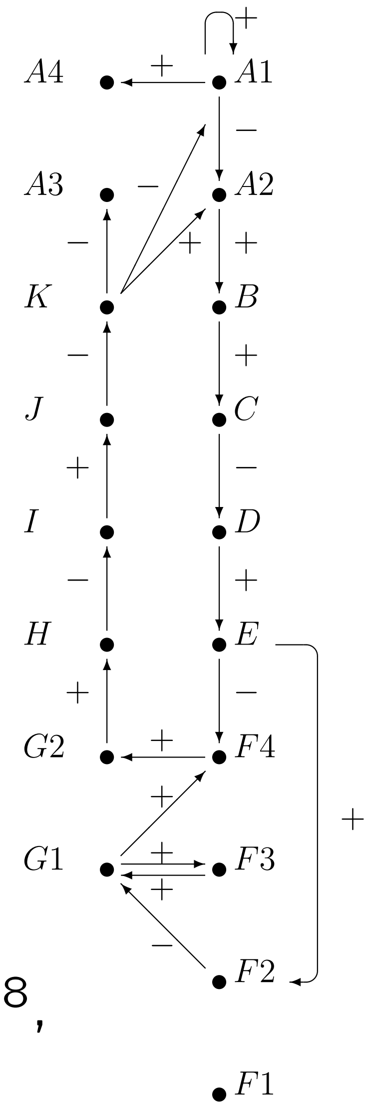
where P_m is a projection onto the first m modes and π_j is a projection onto the jth mode.

This set may be used to compute the index in the infinite-dimensional space.

Chaotic symbolic dynamics for Kot-Schaffer





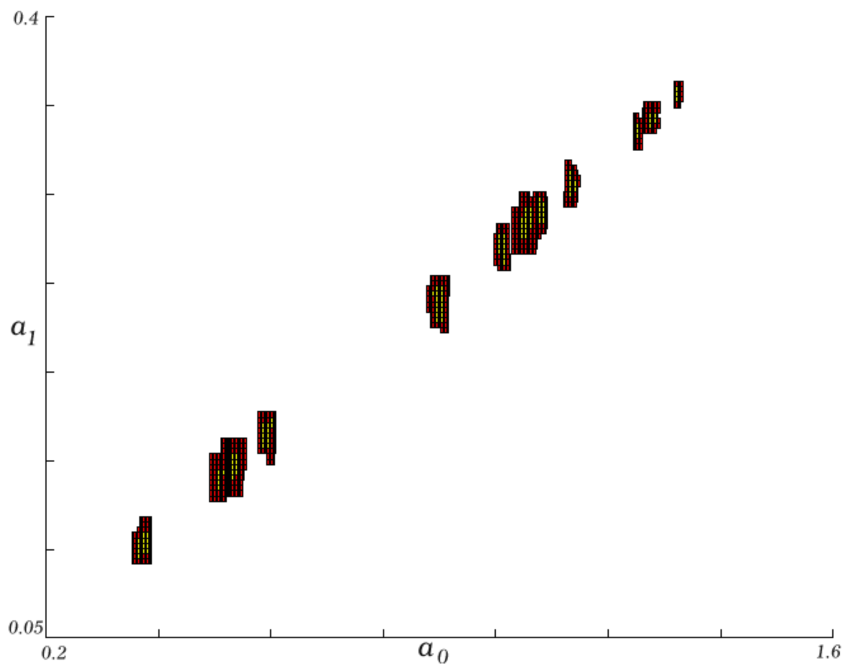


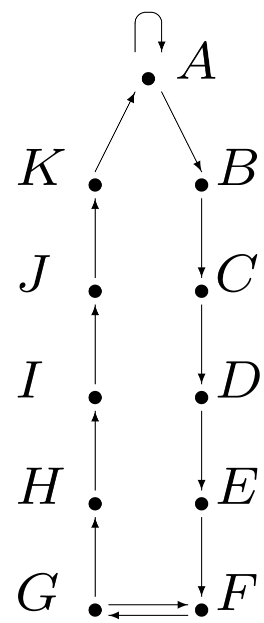
$N = A \cup \ldots \cup K$, $\quad H_2(N_1, N_0) = \mathbb{Z}^{18}$,

$S = Inv(N_1 \setminus N_0, f)$, $\quad f_* = P_m \Phi_*$

Chaotic symbolic dynamics for Kot-Schaffer

Theorem. There is an invariant set given with precision 10^{-10}, which under Φ is semi-conjugate to the chaotic subshift dynamics given by the transition graph shown below.





Proof.

1. Construct a lifting set Z, (GAIO in first 6 modes, analysis in higher modes all using formulas similar to the one given in Lemma 1)

2. A similar argument to the one used for the Hénon map completes the proof.