TOWARDS AUTOMATED CHAOS VERIFICATION*

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We describe a set of algorithms (and corresponding codes) that serve as a basis for an automated proof of existence of a certain dynamical behaviour in a given dynamical system. In particular it is in principle possible to automatically verify the presence of complicated dynamics using these tools. The Hénon map is used to illustrate these techniques.

1. Introduction

The purpose of this note is to give an elementary description of an automated method for proving the existence of low dimensional dynamical objects such as fixed points, periodic orbits, connecting orbits, or even subshift dynamics of finite type for maps. We begin with a discussion of the algorithms used. This includes the combinatorial data structures used by the computer, how to find potential invariant sets with specific dynamic properties, how to refine the approximation, and how to produce computationally useful index pairs. This last point leads us to the Conley index theory [4] which is used to draw rigorous conclusions about the dynamics from the numerical computations. To emphasize the importance of this last point we conclude with an example in which index computations fail to support the existence of a connecting orbit suggested by the numerical computation. Indeed, a much finer numerical approximation shows that such an orbit actually cannot exist.

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2. Data structures and algorithms

We are interested in proving the existence of certain invariant sets of a continuous map $f : X \to \mathbb{R}^n$, where $X \subset \mathbb{R}^n$ is compact. To this end we are going to derive a combinatorial finite state model of the dynamics of f on X in the form of a directed graph, where the states of this model will represent subsets of X.

A *rectangular set* is a subset B of $\mathbb{R}^n$ of the form

$$B = B(c, r) = \{x \in \mathbb{R}^n : |x_k - c_k| \leq r_k, k = 1, \ldots, n\},$$

where $c, r \in \mathbb{R}^n$, $r_k \geq 0$, are the *center* and *radius* of B. Note that by *bisecting* B with respect to the j-th coordinate direction one obtains two rectangular sets $B^- = B(c^-, \hat{r})$ and $B^+ = B(c^+, \hat{r})$, where $\hat{r}_j = r_j/2$ and $c_j^\pm = c_j \pm r_j/2$ and $\hat{r}_k = r_k, c_k^\pm = c_k$ for $k \neq j$. A set which can be represented by iterating this subdivision process, starting with a rectangular set X is a *box*. A *cubical set* is a finite union of boxes. Note that a binary tree represents a certain set of boxes if one assigns a coordinate direction to each level of the tree: the root corresponds to the box X and all nodes of a given level correspond to a subset of a cubical grid on X [2]. Denote by $\mathcal{B}_k$ the collection of all boxes represented by the nodes on level k of a tree (where the root is on level 0). For a subset $\mathcal{B} \subset \mathcal{B}_k$ let $|\mathcal{B}|$ denote the union of all boxes in $\mathcal{B}$. Let $o(\mathcal{B})$ be the set of all boxes in $\mathcal{B}_k$ which intersect $|\mathcal{B}|$, i.e. the smallest representable neighborhood of $|\mathcal{B}|$ in $\mathcal{B}_k$.

For a given box collection $\mathcal{B}$ we are going to represent the dynamics of f by a directed graph $G = (\mathcal{B}, E)$, where we define the set E of edges in G to consist of all pairs (B_0, B_1) such that $f(B_0) \cap B_1 \neq \emptyset$. In order to allow for errors introduced when computing and representing the image $f(B)$ of a box $B \in \mathcal{B}$ on the computer, we actually make use of an *enclosure* of f, i.e. a multivalued map $\mathcal{F} : \mathcal{B} \rightrightarrows \mathcal{B}$ such that $f(B) \subset \operatorname{int} |\mathcal{F}(B)|$ for $B \in \mathcal{B}$. $\mathcal{F}$ can efficiently be represented as e.g. a (sparse) matrix.

2.1. *Finding invariant sets*

The first step in proving the existence of a certain invariant set is the construction of an isolating neighborhood for f which isolates some invariant set of interest. A rough guess for some specific invariant set of f can be obtained efficiently by analyzing the matrix resp. graph $\mathcal{F}$: k-periodic points of f may be present in boxes corresponding to nonzero diagonal entries of $\mathcal{F}^k$ (viewing $\mathcal{F}$ as a matrix), recurrent sets of f should be related to boxes corresponding to strongly connected components of $\mathcal{F}$ (viewing $\mathcal{F}$ as a graph). Connecting orbits can be identified using Dijkstra's shortest path algorithm[3].

Once a guess $\tilde{\mathcal{S}} \subset \mathcal{B}$ for an invariant set has been computed, we construct a combinatorial isolated invariant set $\mathcal{S}$ containing $\tilde{\mathcal{S}}$ by the following procedure. Here $\mathrm{Inv}(\mathcal{S}, \mathcal{F})$ denotes the maximal invariant set of $\mathcal{F}$ within the set $\mathcal{S}$, which can be efficiently computed[7].

Algorithm 1. (Combinatorial isolating neighborhood)

$\mathcal{S}$ = `make_isolated(`$\tilde{\mathcal{S}}$`)`
    $\mathcal{S}$ `:=` $\mathrm{Inv}(\tilde{\mathcal{S}}, \mathcal{F})$
    `while` $o(\mathcal{S}) \not\subset \tilde{\mathcal{S}}$
        $\tilde{\mathcal{S}}$ `:=` $\tilde{\mathcal{S}} \cup o(\mathcal{S})$
        $\mathcal{S}$ `:=` $\mathrm{Inv}(\tilde{\mathcal{S}}, \mathcal{F})$
    `if` $\mathcal{S} \subset \mathrm{int}\,|o(\mathcal{S})|$ `return` $\mathcal{S}$
    `else return` $\emptyset$

Note that once this algorithm returns a nonempty collection $\mathcal{S} =: \mathcal{S}_k \subset \mathcal{B} \subset \mathcal{B}_k$ of boxes, one can efficiently *refine* the isolating neighborhood and obtain a tighter covering of the underlying invariant set by repeatedly (1) subdividing the current set $\mathcal{S}_k$ of boxes, yielding a collection $\mathcal{S}_{k+1}$ and (2) discarding all boxes from $\mathcal{S}_{k+1}$ which are not contained in $\mathrm{Inv}(\mathcal{S}_{k+1}, \mathcal{F})$[2].

2.2. *Computing the index*

Given a cubical isolating neighborhood, the next step is to construct a cubical index pair. We begin with an approach due to Szymczak[7,8]. Given an isolated invariant set $\mathcal{S}$ define $\mathcal{N}_1 := \mathcal{S} \cup \mathcal{F}(\mathcal{S})$ and $\mathcal{N}_0 := \mathcal{N}_1 \setminus \mathcal{S}$. Then, $N = (N_1, N_0) := (|\mathcal{N}_1|, |\mathcal{N}_0|)$ is an index pair for f.

There are two considerations which prompt us to modify this construction. First, index pairs computed in this manner may be very large, especially when studying objects with high dimensional, strongly unstable behavior. Second, computation of the index map on this pair could introduce "folding effects": The computation requires an auxiliary pair which contains the image of the original index pair. However, the definition of the index map demands that the inclusion of the first pair into the second induce an isomorphism in relative homology. This property fails when the image of a box in $\mathcal{N}_0$ folds back to $o(\mathcal{S}) \setminus \mathcal{N}_1$.

In light of these considerations, we propose the following algorithm for computing a *modified combinatorial index pair*. This modification, which is sufficient for defining the Conley index[1], exploits the relationship between the relative homology of the pair (N_1, N_0) and the structure of the quotient space N_1/N_0. Namely, the property of excision allows us to essentially ignore the sets outside of a neighborhood of $N_1 \setminus N_0$. In particular, we construct an index pair in the combinatorial setting which truncates the sets outside of a one box neighborhood of $\mathcal{N}_1 \setminus \mathcal{N}_0$ and, furthermore, avoids folding effects when computing the index map.

Algorithm 2. (Modified Combinatorial Index Pair)

$[\mathcal{P}_1, \mathcal{P}_0]$ `= build_ip(`$\mathcal{S}$`)`
  $\mathcal{P}_0$ `:=` $\emptyset$
  $\mathcal{E}$ `:=` $(\mathcal{F}(\mathcal{S}) \cap o(\mathcal{S})) \setminus \mathcal{S}$
  `while` $\mathcal{E} \neq \emptyset$
    $\mathcal{P}_0$ `:=` $\mathcal{P}_0 \cup \mathcal{E}$
    $\mathcal{E}$ `:=` $(\mathcal{F}(\mathcal{P}_0) \cap o(\mathcal{S})) \setminus \mathcal{P}_0$
  $\mathcal{P}_1$ `:=` $\mathcal{S} \cup \mathcal{P}_0$
  `return` $[\mathcal{P}_1, \mathcal{P}_0]$

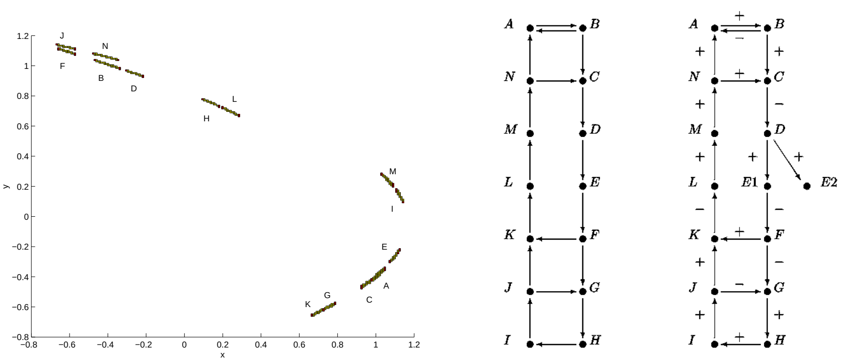
Computation of the index map also requires an auxiliary pair $[\mathcal{Q}_1, \mathcal{Q}_0]$ given by $\mathcal{Q}_1 := \mathcal{F}(\mathcal{P}_1)$ and $\mathcal{Q}_0 := \mathcal{Q}_1 \setminus \mathcal{S}$. Note that by construction, $[\mathcal{P}_1, \mathcal{P}_0] = [\mathcal{Q}_1 \cap o(\mathcal{S}), \mathcal{Q}_0 \cap o(\mathcal{S})]$. Equivalently, $(|\mathcal{P}_1|, |\mathcal{P}_0|) = (|\mathcal{Q}_1| \setminus A, |\mathcal{Q}_0| \setminus A)$ where $A = |\mathcal{Q}_1 \setminus o(\mathcal{S})| \subset |\mathcal{Q}_0|$. By excision, the inclusion map $i : [|\mathcal{P}_1|, |\mathcal{P}_0|] \hookrightarrow [|\mathcal{Q}_1|, |\mathcal{Q}_0|]$ induces an isomorphism i_* in relative homology. This property ensures that we prevent the folding effect and, therefore, may compute the index map.

It is worth noting that given a cubical grid, the pair given by Algorithm 2 is, in many ways, optimal for studying the dynamics on $\mathcal{S}$. Both sets in the pair are contained in $o(\mathcal{S})$, the minimal box collection one should expect to consider when studying the dynamics on $|\mathcal{S}|$. In addition, this pair is just large enough to ensure that folding is avoided in computing the index.

Given a modified combinatorial index pair $[\mathcal{P}_1, \mathcal{P}_0]$ for the map $\mathcal{F}$, we compute its index a software package developed in the Computational Homology Program[5] (see www.math.gatech.edu/∼chom).

3. A numerical example: The Hénon Map

Consider the map $f(x, y) = (1 - ax^2 + y/5, 5bx)$, with parameter values $a = 1.3$ and $b = 0.2$. We apply the previously described procedure in order to prove the existence of an invariant set on which f exhibits complicated dynamics. Based on the results of a simulation with initial point $(0, 0)$ we fix the box X with center $(0.1767, 0.1767)$ and radius $(1.1078, 1.1078)$ as the state space for further investigation. We choose to compute a covering of the maximal invariant set of f up to level 16 of the tree, resulting in a collection of 1188 boxes. We compute guesses for the location of a period two point, a period four point as well as connecting orbits between these two. Applying Algorithms 1 and 2 to this guess results in an isolating neighborhood $\mathcal{S}$ consisting of 14 distinct components and the associated modified combinatorial index pair shown in Figure 1. The homology of this pair is $H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{15},\ 0,\ 0,\ \ldots)$ with the index map $h_{P,1}$ on level one given by the transition graph in Figure 1(c).



(a) (b) (c)

Figure 1. (a) An index pair $[\mathcal{P}_1, \mathcal{P}_0]$ for a guess of two orbits connecting a period 2 orbit and a period 4 orbit for the Hénon map. The 14 components of $\mathcal{S} = \mathrm{cl}\,(\mathcal{P}_1 \setminus \mathcal{P}_0)$ are labeled A through N. (b) The transition graph T representing the multivalued map $\mathcal{H}$ on the 14 components of $\mathcal{S}$. (c) The transition graph (with orientations) representing $h_{P,1}$, the index map on level one. The 15 generators are labeled by their locations in the components of $\mathcal{S}$.

Theorem 3.1. *There is a set contained in $|\mathcal{S}|$, on which f is semi-conjugate to the symbol subshift given by the transition graph T shown in Figure 1(b).*

Proof. Let $\Sigma = \{(Z_i)_{i\in\mathbb{Z}} \mid Z_i \in \{A, B, \ldots, N\}\}$ and let $\sigma : \Sigma \to \Sigma$ be the shift map. Consider the subset, $\Sigma_* = \{(Z_i)_i \in \Sigma \mid (Z_i, Z_{i-1})$ is an edge in $T\}$ which is the closure of periodic orbits given in the transition graph T. Define $\rho : |\mathcal{P}_1| \to \Sigma_*$ by $\rho(x) = (Z_i)_{i\in\mathbb{Z}}$, where Z_i

is the connected component of $|\mathcal{P}_1|$ containing $f^i(x), i \in \mathbb{Z}$. A result of [6] allows us to decompose the index information into maps on the connected components of $|\mathcal{P}_1|$. Let h_{Z*}, $Z \in \{A, \ldots, N\}$, be the index map obtained from $h_{P,*}$ by projecting onto the connected component Z.

If the Lefschetz number of $h_{Z_1*} \circ \ldots \circ h_{Z_n*}$ is not 0, then a periodic orbit under f through the ordered components $Z_1, \ldots, Z_n$ exists. Given that the homology on all other levels is 0, the Lefschetz number is nonzero provided that the trace of the composition of restricted index maps for $H_1(|\mathcal{P}_1|, |\mathcal{P}_0|)$ is nonzero. This can be shown for the composition of restricted index maps for any periodic sequence given by the transition graph in Figure 1(b). By extension, ρ maps onto Σ_*. Therefore, ρ is a semi-conjugacy between f on the (nonempty) invariant set contained in $|\mathcal{S}| = |\mathcal{P}_1| \setminus |\mathcal{P}_0|$ and the shift map σ on Σ_*. □

3.1. *The transition matrix versus algebraic topology*

Using similar techniques to those previously described, we constructed the map $\mathcal{F}$ on a box collection at level 14. The map suggests the existence of a connecting orbit from a period two orbit to a fixed point. However, the associated index computations did not support the existence of this orbit for f. Therefore we repeated the computation at level 24, where the fixed point and the period two orbit (which may be shown to exist by computation of the index at depth 14) persist as expected. However, the map $\mathcal{F}$ itself on the box collection of this level prohibits a connecting orbit.

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