Pitfalls of Mathematical Modelling: The Case of Mortimer Middleman

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Assumption 1: Uniform Demand

Validity versus Tractability
Descriptive versus Prescriptive

Step 1. Decision Variables

r − reorder point (inventory level, when to go to Antwerp)

q − order quantity (how many carats to purchase in Antwerp)

Step 2. Objective or Performance Measure

Minimize Cost: holding cost + replacement cost + lost-sales cost

Step 3. Constraints

$$r \geq 0$$
$$q \geq 100$$

Problem: without additional assumptions we still can't write down a closed form expression for cost.

Assumption 2: No Lost-Sales Allowed

Is this a good idea? Due to the one week lead time on order arrival we know that $r \geq 55$ guarantees no lost-sales.

Let's compare some costs.

The cost of a lost-sale is $200 per carat but the weekly holding cost for a carat is $3.50. Hence, as long as the carat is not held in inventory for more than 57.1 weeks (200/3.5) then we are better off with held inventory than a lost sale.

Now we can write an expression for cost as a function of r and q.

$cost(r, q) =$ holding cost + replinishment cost

Holding Cost (weekly)

average number of carats held * \$3.50

= (high + low)/2 *3.50

$$= \frac{((r - 55) + q) + (r - 55)}{2} * 3.50$$

$$= ((r - 55) + q/2) * 3.50$$

Replenishment Cost (weekly)

cost of replenishment / weeks between replenishments

$$= (\$2000)/(q/55)$$

where $q/55$ is the length of a sawtooth cycle in weeks.

Thus, we can write the performance measure

$$cost(r, q) = ((r - 55) + q/2) * 3.50 + 2000(55/q)$$

We notice immediately that if $r > 55$ that we incur greater holding costs. Thus, to minimize costs and satisfy the constraint $r \geq 55$ we set $r = 55$.

Our resulting single variable optimization problem then is to minimize

$cost(55, q) = 3.50 * (q/2) + 2000 * (55/q)$

subject to the constraint $q \geq 100$. Thus, if we solve the unconstrained problem with single variable calculus and if the global minimum for q is greater than 100 then we are done.

Differentiating $cost(55, q)$ with respect to q

$$3.50/2 - (2000 * 55)/q^2$$

If we set this expression to zero and solve for q we get

$q = \sqrt{\frac{2(2000)55}{3.50}} = 250.7$

Thus, $cost(55, 251) = \$877.50$ per week or \$45,630 per year.

Our predicted cost of \$45,630 compares favorable with MM's actual cost of \$94,009 last year. In fact, we have produced a closed form solution for any problem like this one. The general model is called EOQ (Economic Order Quantity) and has the following solution when assumptions 1 and 2 hold. Let

d = weekly demand
f = fixed replenishment cost
h = holding cost
s = cost of a lost sale
l = lead time
m = minimum order size

$$q^* = \sqrt{\frac{2fd}{h}}$$
$$r^* = ld$$

as long as $q^* \geq m$ and $q^* \geq ld$.

The EOQ model and its solution is an example of a highly *tractable* and *prescriptive* mathematical model. But how good is it? As a partial answer to this question let's turn our attention to *descriptive* models. Consider a simulation model of MM's inventory problem.

Assumption 3: Next Year Will Be EXACTLY Like Last Year

The model is not prescriptive. That is, we must tell it the values for r and q before the simulation will run. The simulation will step through last year's historical data and tell us what would have happened if the values input for r and q had been used.

With $r = 55$ and $q = 251$ the simulated yearly cost is \$108,621 which is far from the EOQ predicted cost of \$45,630 and worse than MM did on his own last year (\$94,009). Clearly this is not good.

How can we effectively use our simulation model (under assumption 3) to make decisions about the appropriate values for r and q? A host of heuristic search procedures are at our disposal (simulated annealing, tabu search, genetic algorithms, evolutionary search etc.). For now we will simply employ a multistart local search procedure. We begin with our EOQ solution. Below is a summary of how such a search might proceed.

(r, q)	cost	step
(55,251)	108,621	
(65,251)	108,421	$r + 10$
(75,251)	63,243	$r + 10$
(85,251)	63,024	$r + 10$
(95,251)	64,242	$r + 10$
(85,261)	95,193	$q + 10$
(85,241	72,781	$q - 10$

A different starting solution yields a local optimum at $cost(135, 261) = 54, 193$.

But how realistic is assumption 3 that next year will be like last year? Perhaps we would be more comfortable if we simply said that *next year will be similar to last year*. In order to quantify this relaxation we let weekly demand be a random variable and plot last year's demand histogram.

Notice that the 40-70 range was most likely last year and we might fit the dashed line as a distribution for demand. (Note there are standard software packages and techniques for fitting data to known distributions.)

Let's simulate the yearly performance of (135,261) and (115,251) and replicate the procedure 200 times.

Discussion

- What were the strong points of the closed form solution?

- What solution should Mortimer select for upcoming years?

- Would this analysis serve as a strategic plan for MM's business?

- What other tools exist for including uncertainty in optimization models?