CSci 688 (Kincaid/MWF 9-9:50 a.m.)

Scale-Free Networks

Sufficient Conditions for Hamiltonian Graphs

A graph G is defined to *hamiltonian* if it has a cycle containing all of the vertices of G. A cycle of G containing every vertex of G is called a *hamiltonian cycle* of G. Because of the similarity in the definitions of eulerian graphs and hamiltonian graphs, and because a particularly useful characterization of eulerian graphs exists, one might well expect an analogous criterion for hamiltonian graphs. However, such is not the case; indeed it must be considered one of the major unsolved problems of graph theory to develop an applicable characterization of hamiltonian graphs.

There have, however, been several sufficient conditions established for a graph to hamiltonian. We present several of these as they developed historically. The hope is to see how a new theorem is developed from an old one. This is one of the ways in which new mathematical ideas are developed. The first sufficiency result appeared in 1952 nearly 100 years after the idea of a hamiltonian graph was put forward.

Theorem H1 (G.A.Dirac). *If G is an order $p \geq 3$ graph such that $\deg v \geq p/2$ for every vertex v of G, then G is hamiltonian.*

Dirac notes that in one sense this is the best possible result. Specifically, the conclusion does not follow if we assume the minimum degree of the vertices to be $(p-1)/2$. Any complete bipartite graph $K_{d,d+1}$ is a counterexample. We also note that the graphs consisting of a single cycle of order five or more show that Dirac's condition is far from necessary for a graph to be hamiltonian.

The next result is a stronger sufficient condition and follows from thinking about a hamiltonian cycle in a different context. Let G be order p and let the vertices of G represent p people who are to be seated around a table. The edges of G correspond to friendships between pairs of individuals. Then the problem of finding a hamiltonian cycle in G corresponds to the problem of seating the p people in such a way that only friends are seated next to each other. In this context Dirac's theorem states that if in a group of p people each person is friends with at least $p/2$ other people in the group, then there exists a seating arrangement of the p people around a table in which everyone is sitting between two friends. Thinking of the problem in this way led to the following theorem in 1960.

Theorem H2 (O.Ore). *If G is an order $p \geq 3$ graph such that for all distinct non-adjacent vertices u and v, $\deg u + \deg v \geq p$, then G is hamiltonian.*

I said that Theorem H2 was a stronger sufficient condition than Theorem H1. Why does H1 imply H2? Find a graph that satisfies H2, but not H1.

Show why (construct a counterexample) H2 is not a necessary condition. The next advance in sufficient conditions came in 1962.

Theorem H3 (L.Posa). *If G is an order $p \geq 3$ graph such that*

(i) for all p and for all $1 \leq j < (p-1)/2$, G has fewer than j vertices of degree j or less, and

(ii) if p is odd, G has no more than $(p-1)/2$ vertices of degree $(p-1)/2$ or less,

then G is hamiltonian.

As before H3 is stronger than H2. Why does H2 imply H3? Find a graph that satisfies H3, but not H2. Show why (construct a counterexample) H3 is not a necessary condition.

Dirac's condition imposes a specific requirement on the degree of each vertex of the graph G. Ore loosened this a bit by imposing a similar requirement on the degrees of certain pairs of vertices. Posa then assumed even less, by restricting the number of vertices that are allowed to have certain small degrees. J. A. Bondy in 1969 was able to weaken the assumption even further by setting a similar limitation on the degrees of certain pairs of vertices. Let $d_1 \leq d_2 \leq \ldots \leq d_p$ be the degrees of the vertices $v_1, v_2, \ldots, v_p$ of an order p graph G.

Theorem H4 (J.A.Bondy). *If G is an order $p \geq 3$ graph satisfying the condition $d_k \leq k$ and $d_m \leq m$ $(k \neq m)$ implies that $d_k + d_m \geq p$, then G is hamiltonian.*

As before H4 is stronger than H3. Why does H3 imply H4? Find a graph that satisfies H4, but not H3. Show why (construct a counterexample) H4 is not a necessary condition. We will end our list of sufficient conditions for hamiltonian graphs here. We note, however, that even stronger sufficient conditions exist. Two more are by Chvatal (1972) and Bondy and Chvatal (1976).