Key for homework 3

Problem 37. *Proposition.* If G is of order p (≥ 2) and every vertex v of G has deg v $\geq (p-1)/2$, then G is connected.

Proof. *(by contradiction)* Assume that G is disconnected and that every vertex v of G has deg $v \geq (p-1)/2$. Then G has k components ($k \geq 2$). Let u_i denote an arbitrary vertex in component G_i of G ($i = 1, \ldots, k$). Since deg $u_i \geq (p-1)/2$, G_i must have at least $1 + (p-1)/2 = (p+1)/2$ vertices. The order of G is the sum of the orders of its components. Hence, we have

$$\begin{aligned}
|V(G)| &= \sum_{i=1}^{k} |V(G_i)| \\
&\geq \sum_{i=1}^{k} (p+1)/2 \\
&= k(p+1)/2 \\
&\geq 2(p+1)/2 \\
&= p+1.
\end{aligned}$$

The first equality is simply a restatement of the definition of the order of a graph and its components. The first inequality follows from the previously stated lower bound on the order of any component G_i of G. The last inequality follows when k is replaced by its smallest value of 2. Thus, we have that the order of G is at least as large as $p+1$. But this is impossible since G is of order p. Hence, G must be connected.

Problem 40. (using Figure 2.14)

a. v_1, v_2, v_4, v_3, v_4, v_2 is a trail (no edge is repeated). It is not a path since vertex v_3 is repeated.

Problem 43. The sequence of vertices and edges given by v_1, v_2, v_3, v_4, v_5, v_2, v_1 is neither a circuit nor a cycle. It cannot be a cycle since vertex v_2 is repeated. It cannot be a circuit since edge v_1v_2 is repeated.