Homework #2
Key, Fall 2009

3-1 4 pts, 3-3 3 pts, 3-4 3 pts, 3-5 3 pts, 3-6 8 pts, 3-8 6 pts, 3-9 3 pts, 3-12 12 pts, 3-15 4 pts, 3-16 3 pts; 49 total

3-1(b) infeasible; feasible (local opt.); feasible; feasible (global opt.)

3-3(a) $\Delta w^{(1)} = w^{(1)} - w^{(0)} = (4,2,6)$; $\Delta w^{(2)} = w^{(2)} - w^{(1)} = (0,-2,12)$; and $\Delta w^{(3)} = w^{(3)} - w^{(2)} = (-1,0,3)$.

(b) $\Delta w^{(1)} = (-7,2,4)$, $\Delta w^{(2)} = (1,4,0)$ and $\Delta w^{(3)} = (0,3,1)$.

3-4 (b) not feasible, not improving (a bit unclear)

(d) improving but not feasible (f) not improving

3-5(b) infeasible (e) infeasible (f) feasible

3-6 (b) Find max. $\lambda \geq 0 \ni (9,4,6) + \lambda(-3,-3,9)$ is feasible. All constraints must hold for $(9-3\lambda, 4-3\lambda, 6+9\lambda)$. Try each one.
$x_1 \geq 0 \Rightarrow 9-3\lambda \geq 0$ or $\boxed{\lambda \leq 3}$. $x_2 \geq 0 \Rightarrow 4-3\lambda \geq 0$ or $\boxed{\lambda \leq \frac{4}{3}}$
$x_3 \geq 0 \Rightarrow 6+9\lambda \geq 0$, true $\forall \lambda \geq 0$. $x_1 - 2x_2 + 3x_3 \leq 25 \Rightarrow (9-3\lambda) - 2(4-3\lambda) + 3(6+9\lambda) \leq 25$ or $30\lambda + 19 \leq 25$ or $\boxed{\lambda \leq 1/5}$. The max. $\lambda \geq 0$ at which <u>all</u> constraints hold is for $\lambda = 1/5$.

(d) as in (b) we check all constraints, $x_1 \geq 0 \Rightarrow 16-4\lambda \geq 0$ or $\boxed{\lambda \leq 4}$.
$x_2 \geq 0 \Rightarrow 2 \geq 0$, true $\forall \lambda$. $x_3 \geq 0 \Rightarrow 1+3\lambda \geq 0$, true $\forall \lambda \geq 0$.
$x_1 - 2x_2 + 3x_3 \leq 25 \Rightarrow (16-4\lambda) - 2(2) + 3(1+3\lambda) \leq 25$ or $\boxed{\lambda \leq 2}$.
Thus, $\lambda = 2$ is the max. value at which <u>all</u> constraints hold.

3-8(b) Min problem so $-\nabla f$ is an improving direction $-\nabla f = (-4,0,-6,5)$.

(d) Max problem so ∇f is an improving direction $\nabla f(2,5) = (3w_1^2 - 4, 6) = (3(2^2) - 4, 6) = (8,6)$.

3-9(b) $(z_1-2)^2 + (z_2-1)^2 \leq 10$ is active at $(5,2)$ since $(5-2)^2 + (2-1)^2 = 9 + 1 = 10$. $2z_1 - z_2 = 8$ is active at $(5,2)$ since $2(5) - 2 = 10 - 2 = 8$. (i) and (ii) are satisfied as equalities and, as a result, are active at $(5,2)$.

3-12(b) itr 1 Both directions are improving. Check feasibility.

$(0,0)+\lambda(2,0) = (2\lambda, 0)$ satisfies all constraints

$$\left.\begin{array}{l} 2(2\lambda)+1(0) \le 9 \\ 0 \le 2\lambda \le 4 \\ 0 \le 0 \le 3 \end{array}\right\} \text{ true for } \lambda>0 \text{ small}$$

$(0,0)+\lambda(-2,4) = (-2\lambda, 4\lambda)$ violates $z_1 \ge 0 \;\forall \lambda > 0$.

Only $(2,0)$ is both feasible and improving. Find $\lambda>0$, the step-size. The constraints restrict λ: $2\lambda \le 4 \Rightarrow \lambda \le 2$, and $2(2\lambda)+1(0) \le 9 \Rightarrow \lambda \le 2\tfrac{1}{4}$. Therefore $\lambda = 2$ is the stepsize, and

$$z^{(1)} = z^{(0)} + \lambda \Delta z$$
$$= (0,0) + 2(2,0) = (4,0)$$

itr 2 Find feasible direction.

$(4,0)+\lambda(2,0) = (4+2\lambda, 0)$ violates $z_1 \le 4 \;\forall \lambda > 0$.

$(4,0)+\lambda(-2,4)$ satisfies all constraints

$$\left.\begin{array}{l} 2(4-2\lambda)+1(4\lambda) \le 9 \\ 0 \le 4-2\lambda \le 4 \\ 0 \le 4\lambda \le 3 \end{array}\right\} \text{ True for small } \lambda>0.$$

Compute stepsize $4\lambda \le 3 \Rightarrow \lambda \le 3/4$; $4-2\lambda \ge 0 \Rightarrow \lambda \le 2$.

Thus $\lambda = 3/4$ simultaneously satisfies all constraints.

$$z^{(2)} = (4,0) + 3/4(-2,4) = (5/2, 3)$$

itr 3 Find feasible direction

$(5/2, 3)+\lambda(2,0) = (5/2+2\lambda, 3)$ satisfies all constraints.

$$\left.\begin{array}{l} 2(5/2+2\lambda)+1(3) \le 9 \\ 0 \le 5/2+2\lambda \le 4 \\ 0 \le 3 \le 3 \end{array}\right\} \text{ true } \forall \lambda>0 \text{ small.}$$

$(5/2, 3)+\lambda(-2,4) = (5/2-2\lambda, 3+4\lambda)$ violates $z_2 \le 3 \;\forall \lambda \ge 0$.

Compute stepsize for $(5/2+2\lambda, 3)$. $2(5/2+2\lambda)+3 \le 9 \Rightarrow \lambda \le 1/4$

$5/2+2\lambda \le 4 \Rightarrow \lambda \le 3/4$. Thus $\lambda = 1/4$ and $z^{(3)} = (5/2,3)+1/4(2,0)$
$= (3,3)$.

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3-12(b) cont.

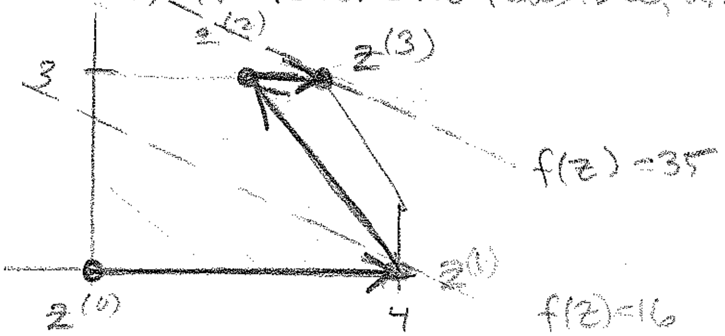
<u>it#4</u> Check feasibility

$(3,3) + \lambda(2,0) = (3+2\lambda, 3)$ violates $2z_1 + z_2 \leq 9 \ \forall \lambda > 0$

$(3,3) + \lambda(-2,4) = (3-2\lambda, 3+4\lambda)$ violates $z_2 \leq 3 \ \forall \lambda > 0$.

Thus, there are no feasible, improving directions. STOP.

(c)



3-15 (a) $\lambda(3,1,0) + (1-\lambda)(0,4,9)$ all points on line segment for $\lambda \in [0,1]$

$z^{(3)} = (2,2,3) = \frac{2}{3}(3,1,0) + \frac{1}{3}(0,4,9)$

Thus, $z^{(3)}$ is on the line segment. $z^{(4)} = (3,5,9)$

For $z^{(4)}$ to be on the line segment

$$(3,5,9) = \lambda(3,1,0) + (1-\lambda)(0,4,9)$$

for some choice of λ. The first component

$3 = 3\lambda + (1-\lambda)(0)$ forces $\lambda = 1$ for equality to hold

But if $\lambda = 1$ then equality cannot hold for the 2nd or 3rd components ($\lambda + (1-\lambda)4 \neq 5$ and $(1-\lambda)9 \neq 9$).

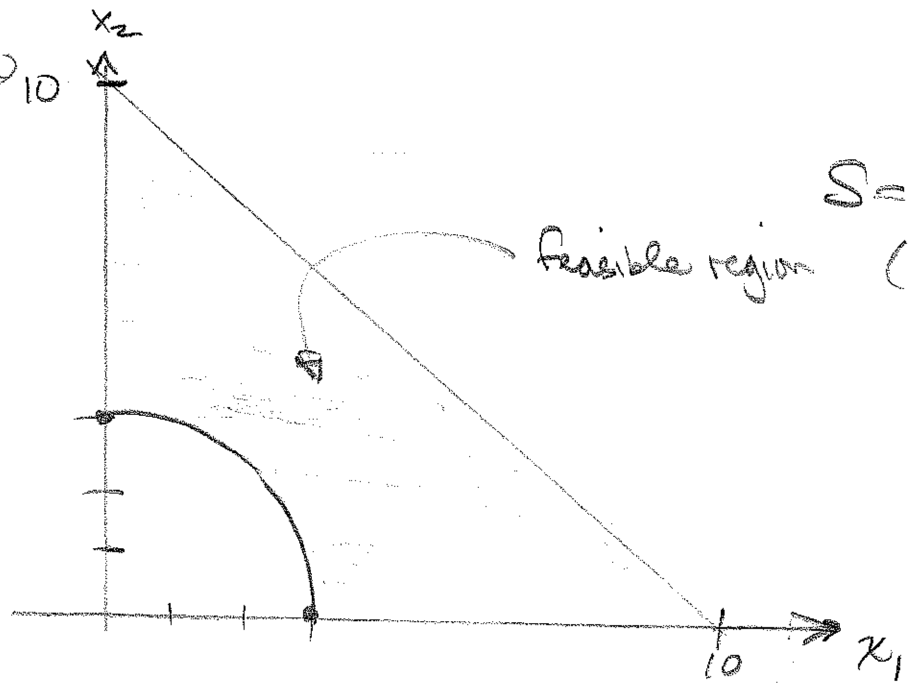
Thus $z^{(4)}$ is not on the line segment

(b) $\lambda(2,1,4) + (1-\lambda)(7,0,-1)$ represent all points on the line segment for $\lambda \in [0,1]$.

$z^{(3)} = (6, \frac{1}{5}, 0) = \frac{1}{5}(2,1,4) + \frac{4}{5}(7,0,-1)$ so $z^{(3)}$ is on the line segment.

For $z^{(4)}$, $(0,3,1) = \lambda(2,1,4) + (1-\lambda)(7,0,-1)$ for some $\lambda \in [0,1]$. But the first component $2\lambda + (1-\lambda)7 = 0$ $\Rightarrow \lambda = 7/5$ which violates $\lambda \in [0,1]$, thus $z^{(4)}$ is not on the line segment.

3-16 (a)



$S = \{ x_1, x_2 \mid (x_1)^2 + (x_2)^2 \geq 9 \cap x_1 + x_2 \leq 10 \cap x_1, x_2 \geq 0 \}$

If S is convex then $\alpha \vec{x}_a + (1-\alpha)\vec{x}_b \in S$ for $\alpha \in [0,1]$ and any $x_a, x_b \in S$.
But if $\vec{x}_a = (0,3)$ and $\vec{x}_b = (3,0)$ then $\alpha(\vec{x}_a) + (1-\alpha)\vec{x}_b \notin S$ for any $\alpha \in (0,1)$. For example, let $\alpha = 1/2$, then $1/2\, \vec{x}_a + \frac{1}{2}\vec{x}_b = (3/2, 3/2)$ and $(3/2)^2 + (3/2)^2 \ngeq 9$.
Thus S is not convex.