CSci 688: Key for homework 1

Problem 18. Page 13. What is the maximum size of an order n graph where n is a positive integer. Everyone did fine on parts a, b, and c. For d, simply writing down the answer $n(n-1)/2$ is not enough. I need to see some explanation or insight into how you found the answer.

Problem 19. Page 13. Everyone did fine on the order 3 graph construction. For the order 4 case many of your explanations fell a bit short.

Does there exist an order 4 graph such that every two vertices are adjacent and every two edges are adjacent.

Assume that such a graph, G, exists. Label the vertices of G as $V = \{v_1, v_2, v_3, v_4\}$. The edges $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$ must exist since v_1 is adjacent to v_2 and v_3 is adjacent to v_4. But, e_1 and e_2 are NOT adjacent (no vertex in common). This is a contradiction and, thus, no order 4 graph exists such that every two vertices are adjacent and every two edges are adjacent.

Problem 20. Page 13. *Proposition.* If G has order $n = |V| \geq 5$ then there exists a graph such that every vertex of G is incident with at least one edge but no two edges are adjacent.

(Note: If the edges of a graph are not adjacent, then each vertex has a degree of at most one. Further, if every vertex is incident with at least one edge then the degree of each vertex must be at least one. Together, these imply that any graph for which the proposition holds consists solely of degree one vertices.)

Case 1.

n is even. In this case $n = 2m$ and V can be partitioned into two equal subsets, V_1 and V_2, with $|V_1| = |V_2| = m$. To construct a graph that meets the conditions of the proposition begin with the sets V_1 and V_2 as previously described and no edges. Select a vertex $v_i \in V_1$ and a vertex $v_j \in V_2$, draw an edge between them. Remove v_i and v_j from further consideration. Repeat this process m times. By construction, every vertex will be incident with exactly one edge and no edges will share a common vertex. $|E| = m$.

Case 2.

n is odd. $n - 1 = 2m$. We can proceed as in Case 1 with $n - 1$ vertices and construct a graph in which each vertex is incident with exactly one edge. This is the fewest number of edges possible. (Clearly if more edges were added the result would be vertices with degree greater than 1 and edges would be adjacent.) To add vertex v_n to the existing graph we must add an edge incident with v_n. However, to do so results in a vertex of degree 2. Thus, two edges must be adjacent which is a contradiction.