MATH 323 (Kincaid/Fall 1996)

Examination 2

November 14

Name: ______________________________

1. The examination is closed book except for one page of 8.5 inch by 11 inch, hand written notes.
2. You may use a calculator, but other than this and your page of notes, you should have only writing instruments at your desk. (No trumpets, trombones, etc.)
3. Write with a soft pencil or a pen (blue or black ink).
4. Write only on the exam sheets provided.
5. Watch the blackboard or overhead for corrections and/or explanations.
6. Read carefully. Problems often have several parts or subsections.
7. The point distribution for each question out of 100 percent is shown in ( ).

1. (16%) *Short answer questions.*

(a) Explain why the dual linear program is infeasible if we know that the primal linear program is unbounded.

(b) We have a maximize linear program. The results of a parametric analysis show that for $\theta = 0$ the objective value $z(\theta) = 276$. As θ increases $z(\theta)$ increases to 336 when $\theta = 4$. Then, as θ increases beyond 4 $z(\theta)$ decreases to 218 when $\theta = 8$. What conclusions can you draw from this information?

(c) What happens to the objective function value of a linear program with a maximize objective function if we replace an $=$ with $\leq$ in a constraint? Why?

(d) Consider a connected, directed network with 79 nodes and 2007 arcs. Explain why the Bellman-Ford shortest path algorithm will never need more than 79 iterations to determine the optimal shortest paths from the source to all other nodes or to determine that a negative dicyle is present?

2. (12%) *Fill in each blank or circle one word as indicated so that each of the following statement is true.*

(a) If the optimal objective value for a primal Phase I linear program is ____________ then original linear program has no feasible solution.

(b) Assume that we have a minimize linear program model. To relax a $\leq$ constraint we (*increase/decrease*) the right hand side.

(c) Assume that we have a minimize linear program model. As we tighten a constraint the impact becomes (*greater/lesser*) the more we relax it.

(d) If, at optimality, a dual variable has a value of non-zero then the slack variable for the associated primal constraint of the linear program model is ($\leq$ /*or* $\geq$ / $=$) 0.

3. (18%) Consider the following *maximize* linear progam. Assume that all main constraints are $\leq$ form and that all decision variables are non-negative.

	x_1	x_2	x_3	x_4	x_5	
c	1	2	0	0	0	
A	-2	1	1	0	0	2
	-1	2	0	1	0	7
	1	0	0	0	1	3
$t = 3$	2nd	1st	N	N	3rd	
$x^{(3)}$	4	1	0	0	2	

Furthermore, you are given that

$$B_3^{-1} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ -2/3 & 1/3 & 0 \\ 2/3 & -1/3 & 1 \end{pmatrix}$$

(a) Compute the reduced costs $\bar{c}_k$ for both nonbasic x_k.

(b) Why is $x^{(3)}$ not optimal?

(c) What is the complementary dual solution (without slacks)? Is it optimal? Why or why not?

4. (18%) For each of the partial simplex searches given below assume that the underlying linear program is a maximize type with $\leq$ main constraints and non-negative decision variables. Further, in each case assume that the origin is feasible. For each simplex search you are asked to determine (i) if an optimal solution has been found; (ii) if an unbounded solution has been reached; (iii) if a degenerate solution has been reached; and (iv) if the complementary dual solution is feasible or infeasible. Circle **all** correct responses.

(a.)

	x_1	x_2	x_3	x_4	x_5
t	B	B	B	N	B
x^{t-1}	10	17	5	0	0
	$v =$	(0,	0,	1,	0)
$\bar{c}_k$	0	0	0	-3	0

(optimal) (primal feasible) (unbounded) (degenerate) (dual feasible) (dual infeasible)

(b.)

	x_1	x_2	x_3	x_4	x_5
t	B	B	N	B	N
x^{t-1}	50	0	0	5	0
		$v =$	(1,	0,	-1/2)
$\bar{c}_k$	0	0	-1	0	1/2
$\Delta x^{(t)}$	1	2	0	5	1

(optimal) (primal feasible) (unbounded) (degenerate) (dual feasible) (dual infeasible)

(c.)

	x_1	x_2	x_3	x_4	x_5
t	B	B	N	N	N
x^{t-1}	3	6	0	0	0
		$v =$	(0,	1/2)	
$\bar{c}_k$	0	0	-1	0	-1/2

(optimal) (primal feasible) (unbounded) (degenerate) (dual feasible) (dual infeasible)

On the previous page is the output obtained for this linear program using LINDO. Using this output and your intimate knowledge of linear programs, answer the following questions.

(a.) (6 %) By how much will the optimal cost change if the minimum daily requirement on chocolate is decreased from 6 ounces to 4 ounces? increased from 6 ounces to 10 ounces.?

(b.) (4 %) Currently brownies are not used as an input to produce the minimum cost diet. At what cost will brownies become an attractive input to produce a minimum cost diet?

(c.) (5 %) State the dual of this linear program.

5. (21 %) My diet requires that all the food I eat come from one of the four *basic food groups*—chocolate cake, ice cream, soda, and cheesecake. At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and cherry cheesecake. Each brownie costs 50 cents, each scoop of chocolate ice cream costs 20 cents, each bottle of cola costs 30 cents, and each piece of cherry cheesecake costs 80 cents. Each day, I must ingest at least 500 calories, 6 ounces of chocolate, 10 ounces of sugar, and 8 ounces of fat. The nutritional content per unit of each food is shown in the table below.

FOOD	Calories	Choc. (oz.)	Sugar (oz.)	Fat (oz.)
Brownie (one)	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Chy. cheesecake (1 piece)	500	0	4	5

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MIN      50 X1 + 20 X2 + 30 X3 + 80 X4
SUBJECT TO
        2)    400 X1 + 200 X2 + 150 X3 + 500 X4 >=    500
        3)    3 X1 + 2 X2 >=    6
        4)    2 X1 + 2 X2 + 4 X3 + 4 X4 >=    10
        5)    2 X1 + 4 X2 + X3 + 5 X4 >=    8
END

LP OPTIMUM FOUND AT STEP       2

       OBJECTIVE FUNCTION VALUE

       1)       90.000000

VARIABLE        VALUE          REDUCED COST
      X1         .000000         27.500000
      X2        3.000000           .000000
      X3        1.000000           .000000
      X4         .000000         50.000000

     ROW   SLACK OR SURPLUS     DUAL PRICES
      2)      250.000000          .000000
      3)         .000000         2.500000
      4)         .000000         7.500000
      5)        5.000000          .000000

NO. ITERATIONS=       2

RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
VARIABLE        CURRENT        ALLOWABLE        ALLOWABLE
                 COEF          INCREASE         DECREASE
      X1       50.000000        INFINITY        27.500000
      X2       20.000000       18.333330         5.000000
      X3       30.000000       10.000000        30.000000
      X4       80.000000        INFINITY        50.000000

                          RIGHTHAND SIDE RANGES
     ROW        CURRENT        ALLOWABLE        ALLOWABLE
                  RHS          INCREASE         DECREASE
       2      500.000000      250.000000         INFINITY
       3        6.000000        4.000000         2.857143
       4       10.000000        INFINITY         4.000000
       5        8.000000        5.000000         INFINITY
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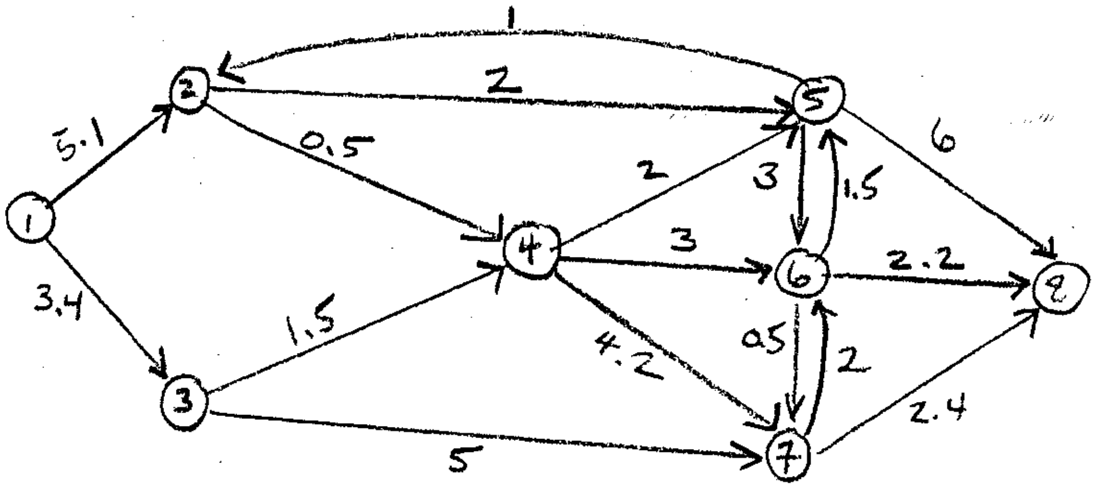
(d.) (9 %) Suppose a new food, chocolate covered raisins, is available at a cost of 25 cents per serving. A serving of chocolate covered raisins has a nutritional content of 350 calories, 2 ounces of chocolate, 2 ounces of sugar, and 1 ounce of fat. Will this new food affect the current optimal diet? Why or why not?

6. (15 %) Consider the undirected graph given below. The table records part of the first three iterations of the Bellman-Ford algorithm with node 1 as the source. Answer the following questions concerning this algorithm.



itr	$k =$	1	2	3	4	5	6	7	8
t=0	$v^{(0)}[k]$	0	∞	∞	∞	∞	∞	∞	∞
t=1	$v^{(1)}[k]$	0	5.1	3.4	∞	∞	∞	∞	∞
	d[k]	–	1	1	1	–	–	–	–
t=2	$v^{(2)}[k]$	0	4.4	3.4	4.9	7.1	∞	8.4	∞
	d[k]	–	2	1	3	2	–	3	–
t=3	$v^{(3)}[k]$	0	4.4	3.4	4.9			8.4	10.8
	d[k]	–	2	1	3			3	7

(a.) (6 %) Compute $v^{(3)}[5]$ and $v^{(3)}[6]$. Show all your work!

(b.) (2 %) What does $d[7] = 3$ at iteration $t = 2$ tell you?

(c.) (4 %) After iteration 3, what is the shortest path from 1 to 8? At what future iteration will this shortest path change? Why?

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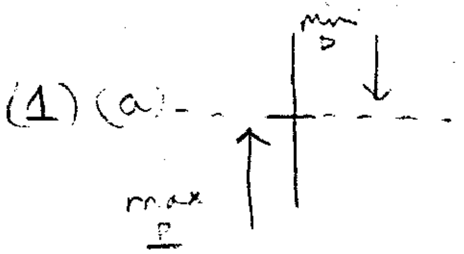
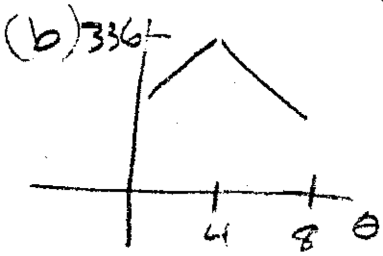
(1) (a) 

Assume primal is a max. Primal and dual solutions come in complementary pairs. In each pair that has a primal feas. solution the dual is primal infeasible except at optimality. Since $c^T x$ increases w/o bound each complementary pair will contain a dual infeasible solution. In addition it is not possible for the dual to bound the primal $\exists$ no $\bar{v} \ni c^T x \leq b^T \bar{v}$ for all feasible x.

(b) 

The tradeoffs described by our parametric analysis are maximized when $\theta = 4$. The rate of increase in $Z(\theta)$ is 15 per unit increase in θ $\forall \theta \in (0,4)$. The rate of decrease in $Z(\theta)$ is $29\frac{1}{2}$ $\forall \theta \in (4,8)$.

(c) Feasible region is enlarged or stays the same (relaxed) and the optimal solution will improve or stay the same.

(d) In a 79 node graph the max # of edges in any path is 78. Thus, if the shortest path values are changing at iteration 79 this means that a 79 edge path is used and the path revisits a previous node. This means that a directed cycle is present and the only directed cycle that would decrease the current shortest path values is one whose cost is < 0. That is, we have a negative dicycle.

2) (a) nonzero (b) increase (c) greater (d) =

3) (a) $\bar{c}_N = c_N - c_B B^{-1} N = (4/3, -5/3)$

(b) $\bar{c}_3 = 4/3 > 0$ so x_3 can enter the basis and improve the obj. val.

(c) $v = c_B B^{-1} = (-4/3, 5/3, 0)$ – no slacks

4) (a) optimal, primal feasible, degenerate, dual feasible

(b) primal feasible, unbounded, degenerate, dual infeasible

(c) optimal, primal feasible, dual feasible

5 (a) Valid range for current basis to remain optimal is $[3\frac{1}{7}, 10]$ and the associated dual variable is $v_2 = 2.50$.
If $6 \to 4$ the optimal cost decreases by $2 \cdot (2.50) = 5$.
If $6 \to 10$ the optimal cost increases by $4(2.50) = 10$

(b) Brownies currently cost 50¢. By decreasing the cost by $27\frac{1}{2}$¢ to $22\frac{1}{2}$¢ brownies will enter the basis.

(c)
$$\text{Max } 500v_1 + 6v_2 + 10v_3 + 8v_4$$
$$\text{s.t. } 400v_1 + 3v_2 + 2v_3 + 2v_4 \le 50$$
$$200v_1 + 2v_2 + 2v_3 + 4v_4 \le 20$$
$$150v_1 + 4v_3 + v_4 \le 30$$
$$500v_1 + 4v_3 + 5v_4 \le 80$$
$$v_1, v_2, v_3, v_4 \ge 0$$

(d) New dual constraint is
$$350v_1 + 2v_2 + 2v_3 + v_4 \le 25$$
$$350(0) + 2(2.5) + 2(7.5) + 1(0) \le 25$$
$$20 \le 25$$

Since the dual constraint is satisfied by current v^* the current basis remains optimal.

6 (a) $v^3[6] = \min\{v^{(2)}[4]+3, v^{(2)}[7]+2, v^{(2)}[5]+3\}$
$= \min\{7.9, 10.4, 10.1\} = 7.9$

$v^3[5] = \min\{v^{(2)}[2]+2, v^{(2)}[6]+1.5, v^{(2)}[4]+2\}$
$= \min\{6.4, +\infty, 6.9\} = 6.4$

(b) It tells us that node 3 is the immediate predecessor node to node 7 on the shortest path from the source to node 7 using at most 2 edges.

(c) 1, 3, 7, 8 is the path at iteration 3. At iteration 4 there is one 4 edge path 1, 3, 4, 6, 8 of length less than 10.8.