cop 2

Math 323 (Kincaid, Fall 1993)

Exam 2

Name: Key

1. (16%) *Short answer questions.*

(a) If the optimal objective function value at the conclusion of a Phase I linear program is nonzero does the original linear program have a feasible solution? Why or Why not?

No it does not have a feasible solution. Why? There is an artificial variable with a nonzero value implying that some constraint cannot be met.

(b) What does $\bar{c}_k = c_k - \mathbf{v} \cdot \mathbf{a}^{(k)}$ for any nonbasic variable k tell us?

$\bar{c}_k$ are the reduced costs and dual slacks. It tells us the per unit change in the primal objective function if x_k were to enter the basis.

(c) What happens to the objective function value of a linear program with a minimize objective function if we replace a $\leq$ or $\geq$ with $=$ in a constraint? Why?

The feasible region is decreased in size (tightened). Hence, the objective function value increases or remains the same.

(d) What is the maximum number of iterations required for the Bellman-Ford shortest path algorithm to determine the optimal shortest paths from the source to all other nodes or to determine that a negative dicyle is present? Explain your answer.

the max number of iterations is = the number of vertices, t. If no $v^{(t)}[k]$ value changes from the previous iteration, t-1, then we have the optimal paths. otherwise, a negative dicycle is present

2. (12%) *Fill in each blank or circle one word as indicated so that each of the following statement is true.*

(a) If a primal linear program is unbounded, then its corresponding dual linear program is infeasible.

(b) Assume that we have a maximize linear program model. To tighten a $\leq$ constraint we (*increase*/[✓ *decrease*]) the right hand side.

(c) Assume that we have a maximize linear program model. As we relax a constraint the impact becomes (*greater*/[✓ *lesser*]) the more we relax it.

(d) If, at optimality, a constraint of a linear program model is inactive, then the corresponding dual variable is ($\leq$ *or* $\geq$ [✓ $=$])0.

3. (18%) Consider the following *maximize* linear progam. Assume that all main constraints are $\leq$ form and that all decision variables are non-negative.

	x_1	x_2	x_3	x_4	x_5	
c	3	2	0	0	0	
	1	0	1	0	0	12
A	1	3	0	1	0	45
	2	1	0	0	1	30
$t = 3$	1	3	2	N	N	
$x^{(3)}$	9	12	3	0	0	

Furthermore, you are given that

$$B_3{}^{-1} = \begin{pmatrix} 0 & -1/5 & 3/5 \\ 1 & 1/5 & -3/5 \\ 0 & 2/5 & -1/5 \end{pmatrix}$$

(a) Compute the reduced costs $\bar{c}_k$ for all nonbasic x_k.

$$v = c_B B^{-1} = (3\,0\,2)\begin{pmatrix} 0 & -1/5 & 3/5 \\ 1 & 1/5 & -3/5 \\ 0 & 2/5 & -1/5 \end{pmatrix} = (0, 1/5, 7/5)$$

$$\bar{c}_4 = 0 - (0, 1/5, 7/5)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1/5$$

$$\bar{c}_5 = 0 - (0\ 1/5\ 7/5)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -7/5$$

(b) Is $x^{(3)}$ optimal? Why or why not?

Yes $x^{(3)}$ is optimal. $\nexists$ an improving direction. All $\bar{c}_k \leq 0$. Local optima are global optima for LPs.

(c) What is the complementary dual solution? Is it optimal? Why or why not?

$v = (0, 1/5, 7/5, 0, 0)$

Strong duality tells us that this is an optimal dual solution.

4. (18%) For each of the partial simplex searches given below assume that the underlying linear program is a maximize type with $\leq$ main constraints and non-negative decision variables. Further, in each case assume that the origin is feasible. For each simplex search you are asked to determine (i) if an optimal solution has been found and, if so, do alternate optimal solutions exist; (ii) if an unbounded solution has been reached; (iii) if a degenerate solution has been reached; and (iv) if the complementary dual solution is feasible or infeasible. Circle **all** correct responses.

(a.)

	x_1	x_2	x_3	x_4	x_5
t	B	B	N	B	N
x^{t-1}	10	0	0	5	0
		$v =$	(1,	0,	1/2)
$\bar{c}_k$	0	0	-1	0	-1/2

[✓ (optimal)] (alternate optimal) (unbounded) [✓ (degenerate)] [✓ (dual feasible)] (dual infeasible)

(b.)

	x_1	x_2	x_3	x_4	x_5
t	B	B	B	N	B
x^{t-1}	50	50	0	0	5
	$v =$	(0,	0,	-1,	0)
$\bar{c}_k$	0	0	0	1	0
$\Delta x^{(t)}$	1/2	1/2	0	1	0

(optimal) (alternate optimal) [✓ (unbounded)] [✓ (degenerate)] (dual feasible) [✓ (dual infeasible)]

(c.)

	x_1	x_2	x_3	x_4	x_5
t	B	B	N	N	N
x^{t-1}	10	5	0	0	0
		$v =$	(0,	1/2)	
$\bar{c}_k$	0	0	-1	0	-1/2

[✓ (optimal)] [✓ (alternate optimal)] (unbounded) (degenerate) [✓ (dual feasible)] (dual infeasible)

5. (21 %) Consider a manufacturer of electronic devices who markets two measuring instruments, XACT and CLOSE, at \$12 and \$15, respectively. The instruments are similar in design and differ only in the numbers of various components which are needed for their assembly. The Bills of Material for the two instruments are as follows:

Product	**XACT**	**CLOSE**
Component a	6 per unit	2 per unit
Component b	15 per unit	12 per unit
Component c	4 per unit	10 per unit
Component d	3 per unit	5 per unit

Let us assume further that the manufacturer is producing for inventory, so that she has a free choice whether to assemble XACT, or CLOSE, or both. Because of their similarity, the two instruments have the same labor costs and overhead expense. In deciding what to produce for a given, the manufacturer need only consider her inventory of components. The following schedule shows how many components of each type are available and their unit costs:

	# in stock	cost per unit
Component a	180	\$ 0.10
Component b	720	\$ 0.05
Component c	400	\$ 0.50
Component d	210	\$ 0.80

A linear program has been formulated to determine the daily production schedule which will earn the maximal gross profit for the manufacturer. Let X denote the number of units of XACT to make per day and Y denote the number of units of CLOSE to make per day. The linear program then is

$$
\begin{aligned}
\max\ & 6.25X + 5.20Y + c_3 Z \\
& 6X + 2Y \leq 180 \\
& 15X + \overset{12}{\cancel{10}}Y \leq 720 \\
& 4X + 10Y \leq 400 \\
& 3X + 5Y \leq 210 \\
& X \geq 0, Y \geq 0
\end{aligned}
$$

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            OBJECTIVE FUNCTION VALUE

            1)      281.00000

      VARIABLE          VALUE              REDUCED COST
             X          20.000000               .000000
             Y          30.000000               .000000


           ROW    SLACK OR SURPLUS        DUAL PRICES
            2)           .000000              .652083
            3)        120.000000              .000000
            4)         20.000000              .000000
            5)           .000000              .779167

     NO. ITERATIONS=         2

     RANGES IN WHICH THE BASIS IS UNCHANGED:

                                  OBJ COEFFICIENT RANGES
     VARIABLE           CURRENT        ALLOWABLE          ALLOWABLE
                          COEF          INCREASE           DECREASE
            X          6.250000         9.349999           3.130000
            Y          5.200000         5.216667           3.116667

                                  RIGHTHAND SIDE RANGES
          ROW           CURRENT        ALLOWABLE          ALLOWABLE
                          RHS           INCREASE           DECREASE
            2        180.000000        64.000000          48.000000
            3        720.000000         INFINITY         120.000000
            4        400.000000         INFINITY          20.000000
            5        210.000000         9.230769         120.000000
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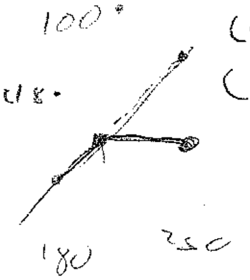
Above is the output obtained for this linear program using LINDO. Using this output and your intimate knowledge of linear programs, answer the following questions.

(a.) (6 %) By how much will the optimal profit change if the # in stock for component a is decreased to 170? increased to 280?



(i) 180 ↓ to 170 Δ in optimal profit = $-10 * .652 = -6.52$ \$

(ii) 180 ↑ to 280 Δ in optimal profit $\leq 100 * .652 = 65.20$

range ends at 216.923, v_1^* decreases.

(b.) (4 %) By how much will the profit per unit of XACT need to increase before the optimal solution changes?

decrease

range for obj. coeff. is [3.12, 15.60] over which opt. soln. is unchanged. Hence, 6.25 - 3.12 = 3.13 is the max decrease. $\Delta > 3.13$.

Max increase is 9.35

(c.) (3 %) Which of the four components is the most valuable to us (ie. for which component would we be willing to spend the most money to purchase additional units)? Why?

Component d is most valuable (per unit) Marginal increase is \$.779 per unit. Is largest.

(d.) (4 %) Suppose that we can retool Component a's and create additional Component d's. For each new Component d we create we must use up 2 Component a's. Show how LINDO can be used to study (parametrically) the effect of making Component d's from Component a's.

$$\begin{aligned} 6X + 2Y &\leq 180(1-2\theta) \\ 3X + 5Y &\leq 210(1+\theta) \\ \theta &= b_5 \end{aligned} \Rightarrow \begin{aligned} 6X + 2Y + 360\theta &\leq 180 \\ 3X + 5Y - 210\theta &\leq 210 \\ \theta &= b_5 \end{aligned}$$

+ original constraints

where b_5 is varied between 0 and 1

θ is a new (unrestricted) decision variable (e.) (4 %) Suppose the manufacturer decides to introduce a new measuring device PRECISE. The new device requires 10 units of component a, 5 units of component b, 3 units of component c, and 10 units of component d. At what profit level will PRECISE change the current optimal solution?

New dual constraint is

$10V_1 + 5V_2 + 3V_3 + 10V_4 \geq C_3$

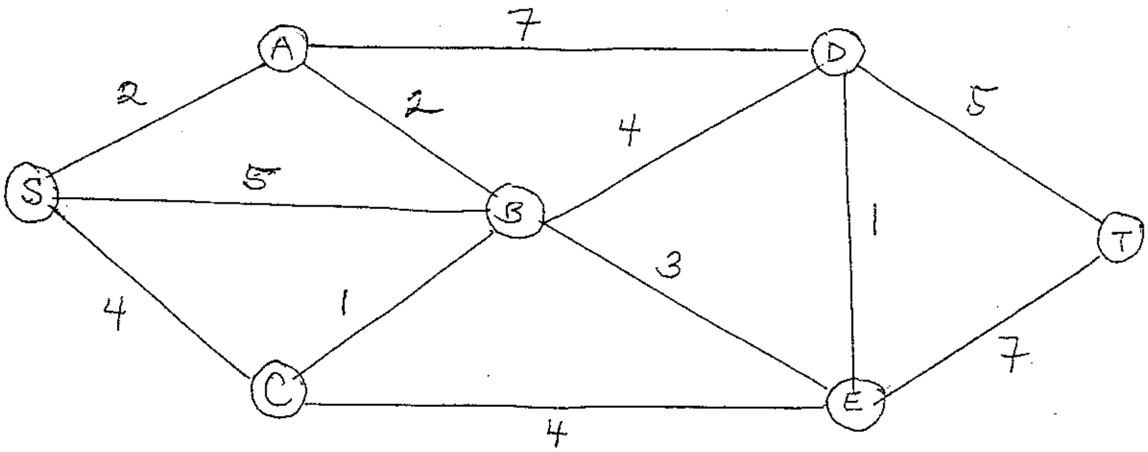
If dual constraint remains PRECISE will not affect optimal soln.

$10(.652) + 0 + 0 + 10(.779) = 14.31 \overset{?}{\geq} C_3$

if $C_3 > 14.31$ then opt. soln. will change

Cost to make PRECISE is $10(.1) + 5(.05) + 3(.5) + 10(.8) = 11.$

6. (15 %) Consider the undirected graph given below. The table records the first four iterations of the Bellman-Ford algorithm with node S as the source. Answer the following questions concerning this algorithm.



itr		k=S	k=A	k=B	k=C	k=D	k=E	k=T
t=0	$v^{(0)}[k]$	0	∞	∞	∞	∞	∞	∞
t=1	$v^{(1)}[k]$	0	2	5	4	∞	∞	∞
	d[k]	–	S	S	S	–	–	–
t=2	$v^{(2)}[k]$	0	2	4	4	9	8	∞
	d[k]	–	S	A	S	A,B	B,C	–
t=3	$v^{(3)}[k]$	0	2	4	4	8	7	14
	d[k]	–	S	A	S	B	B	D
t=4	$v^{(4)}[k]$	0	2	4	4	8	7	13
	d[k]	–	S	A	S	B, E	B	D

$$v^{(4)}[T] \overset{\min}{=} \{ v^{(3)}[D] + C_{DT} ; v^{(3)}[E] + C_{ET} \}$$

$$= \min \{ 8+5 , 7+7 \} = 13$$

(a.) (4 %) Compute $v^{(4)}[T]$. Show all your work!

$$v^{(3)}[D] = \min \{ v^{(2)}[B] + C_{BD} ; v^{(2)}[A] + C_{AD} , v^{(2)}[E] + C_{ED} , v^{(2)}[T] + C_{TD}]$$

$$= \min \{ 4+4 , 2+7 , 8+1 , \infty \}$$

$$= 8$$

(c.) (4 %) Complete iteration 4 in the above table. Show any work needed to complete table here. Details are not as important here as in part (b.).

see table

(d.) (3 %) What does $d[D] = A, B$ at iteration $t = 2$ tell you?

predecessors of node D is A and B
(2 shortest paths)

(e.) (4 %) Assume that iteration 4 is the final one. What is the shortest path from S to T? Is there more than one shortest path? Why or Why not?

T-D-B-A-S
T-B-E-B-A-S