Robust network sensor location for complete link flow observability under uncertainty

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The link observability problem is to identify the minimum set of links to be installed with sensors that allow the full determination of flows on all the unobserved links. Inevitably, the observed link flows are subject to measurement errors, which will accumulate and propagate in the inference of the unobserved link flows, leading to uncertainty in the inference process. In this paper, we develop a robust network sensor location model for complete link flow observability, while considering the propagation of measurement errors in the link flow inference. Our model development relies on two observations: (1) multiple sensor location schemes exist for the complete inference of the unobserved link flows, and different schemes can have different accumulated variances of the inferred flows as propagated from the measurement errors. (2) Fewer unobserved links involved in the nodal flow conservation equations will have a lower chance of accumulating measurement errors, and hence a lower uncertainty in the inferred link flows. These observations motivate a new way to formulate the sensor location problem. Mathematically, we formulate the problem as min–max and min–sum binary integer linear programs. The objective function minimizes the largest or cumulative number of unobserved links connected to each node, which reduces the chance of incurring higher variances in the inference process. Computationally, the resultant binary integer linear program permits the use of a number of commercial software packages for its globally optimal solution. Furthermore, considering the non-uniqueness of the minimum set of observed links for complete link flow observability, the optimization programs also consider a secondary criterion for selecting the sensor location scheme with the minimum accumulated uncertainty of the complete link flow inference.

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1. Introduction

1.1. Research subject and motivation

The importance of link flow information for traffic management and the financial burden of installing sensors bring forth the need to effectively estimate all link flows from limited measurements in a network. The link flow observability problem...
is to identify the minimum number of sensors and their optimal locations such that the measurements collected allow the full determination of flows on all the unobserved links (Castillo et al., 2013b, 2015). The minimum number of sensors guarantees the lowest installation and operation cost, while the optimal locations ensure full observability, i.e., all unobserved link flows can be inferred from the observed flows. The full determination of all link flows is useful for numerous transportation applications, such as origin-destination (OD) trip table estimation, path flow estimation, network performance assessment, evacuation planning, and pavement management (see, e.g., Hu et al., 2009; Ng, 2012, 2013 and the references therein).

In reality, the observed link flows are inevitably subject to measurement error/inaccuracy. For example, Caltrans performance measurement system (PeMS) is a statewide repository for traffic data gathered by thousands of automatic sensors. Bickel et al. (2007) discussed detecting sensor malfunction, imputation of missing or bad data, estimation of velocity, and forecasting of travel times on freeway networks. They pointed out single-loop detectors are the main source of traffic data in California, but loop data are often missing or invalid due to communication error or hardware breakdown. A loop detector can fail in many ways even when it reports values. Payne et al. (1976) identified various types of detector errors, such as stuck sensors, hanging on or hanging off, chattering, cross-talk, pulse breakup, intermittent malfunction, etc. Even under normal conditions, loop detector measurements could be noisy, e.g., due to the confusion of multi-axle trucks. These measurement errors are not constrained to be localized. Instead, they can accumulate and propagate to affect the uncertainty of unobserved link flows, which subsequently can reduce the reliability of link flow inference. Other than the source of measurement errors, the variability or uncertainty of the observed link flows also adds to the problem. In general, the measurements represent just a sample of the variable link flows. Using these sample measurements to infer the unobserved link flows adds to the uncertainty of the estimates. However, for both types of uncertainty sources, the location-specific degree of uncertainty for the observed link flows is unavailable before specifying the observed links and measuring their traffic counts. Our study focus is a new network/region, where there is no existing sensor and measurement information. Hence, an explicit uncertainty consideration in the optimization of traffic sensor locations for complete link flow observability is particularly crucial at the strategic planning stage of sensors installation, where observed link flows and their measurement errors are not yet available. We still can indirectly/implicitly consider the uncertainty propagation based on network topology when optimizing the traffic sensor locations without prior knowledge on the variance. This consideration will reduce the uncertainty in the unobserved link flows, while still ensuring the complete link flow observability in sensors location selection.

1.2. Classification of network sensor location problem

Gentili and Michandani (2012) provided a detailed review on the network sensor location problem (NSLP) by categorizing it into two main types: sensor location flow-observability problem and sensor location flow-estimation problem. The former identifies the optimal placement of sensors that allows the unique determination of the unobserved flows based on the system of linear equations associated with the sensors. The latter identifies the optimal placement of sensors to best improve the quality of related estimates (e.g., OD demands, link/route flows) obtained by the system of linear equations associated with the sensors. Fig. 1 provides a selective summary of these two research problems with a focus on the uncertainty consideration, thus providing a comprehensive review is not the purpose of this study.

For the sensor location flow-estimation problem, majority of the literature focused on the link counting locations for OD estimation. Yang et al. (1991) first proposed the concept of maximal possible relative error to analyze the reliability of an OD matrix estimated from traffic counts. Yang and Zhou (1998) further derived four location rules: OD covering rule, maximal flow fraction rule, maximal flow-intersecting rule, and link independence rule. EhIert et al. (2006) adopted the OD covering rule to develop the second best solution to locate additional counts with budget consideration. Yang et al. (2006) proposed the OD separation rule for the screenline-based traffic counting location problem. Chootinan et al. (2005b) considered the bi-objective counting location problem: the minimum number of counts to separate all OD pairs, and the maximal coverage for a given number of counts. Chen et al. (2007) adopted the OD separation rule and developed strategies for selecting additional counts for improving OD estimation. All of the above studies assumed that the traffic counts are error-free.

The uncertainty in sensor location flow-estimation problem has multiple sources, e.g., prior OD demand variability and measurement errors of traffic counts. Among others, Fei et al. (2007) and Fei and Mahmassani (2011) used the Kalman filtering method to identify a set of sensor locations that optimize the OD demand coverage and maximize the information gains through the observed data, while allowing for measurement errors and minimizing the uncertainties of the estimated OD demands in a time-varying context. Zhou and List (2010) optimized the locations of traffic counting stations and automatic vehicle identification (AVI) readers by maximizing the expected information gain for the OD estimation problem. Several error sources were considered, such as the uncertainty in historical demand information, measurement errors, and approximation errors of link flow proportions. The variability of the posterior OD estimate was measured through the trace of the covariance matrix. A linear measurement equation with an error term is used to relate the unknown OD demand to both point (e.g., link counts) and point-to-point (e.g., vehicle identification counts) measurements, where the distribution and covariance of measurement errors were explicitly considered. Simonelli et al. (2012) developed a network sensor location procedure by using a synthetic dispersion measure to quantify the variability of the posterior OD matrix estimate. A linear measurement equation under the error-free assumption was used to relate travel demand vector and link flow vector. Wang et al. (2012) maximized the variance reductions in posterior route flow estimates while considering the prior route flows and their reliabilities. Wang and Michandani (2013) optimized sensor locations using Bayesian inference to
minimize the uncertainties in route flow estimates while considering the reliabilities of prior route flows and link measurements. Following Zhou and List (2010) and Simonelli et al. (2012), Zhu et al. (2014) also adopted the trace of the covariance matrix of the posterior traffic flow estimation (including OD flows and unobserved link flows) as a measure of variability in the NSLP. These studies also belong to the sensor location flow-estimation problem but with uncertainty consideration.

As to the sensor location flow-observability problem, we focus on the complete link flow observability. Hu et al. (2009) addressed the network sensor location problem (NSLP) from an assumption-free perspective (i.e., no assumption on prior knowledge such as availability of turning proportions at an intersection, link use proportions, and route choice behavioral rules). This is entirely different from the traditional way of treating the NSLP as a sub-problem in some broader applications (e.g., OD estimation, travel time estimation). Instead, they developed a basis-link approach by using the link-path incidence matrix to represent the network structure and then identifying its basis in a matrix algebra context. Castillo et al. (2010b) provided some matrix tools for some general observability problems (including the link flow observability problem) in terms of the link-path incidence matrix. To tackle the path enumeration issue, Ng (2012) proposed a node-based approach using the node-link incidence matrix. Ng (2013) further extended the node-based approach to the partial link flow observability problem in the presence of initial sensors. Castillo et al. (2013a, 2014) refined the upper bound of the number of sensors for the complete link flow observability using the linearly independent (L.I.) paths. The above studies belong to the algebraic approach. Along a different line, He (2013) developed a graphical approach by making use of the topological tree-shaped characteristic. Other than link counts, the NSLP was also studied in the presence of plate scanning technique for OD and route flow observability (e.g., Castillo et al. 2008, 2013b; Miguez et al., 2010). Particularly, Castillo et al. (2010a) considered scanning errors when locating vehicle-ID sensors to observe and estimate route flows by allowing for a partial replication of scanned links (i.e., more than one sensor on the same link). To reduce the probability of scanning errors, the number of scanned links per route is required to be small. Also, Rinaldi et al. (2013) and Viti et al. (2014) provided information theory-based methods and a null-space metric to quantify the solution quality of the partial observability problem when missing information can occur. Yang and Fan (2015) established a connection between the O-D observability analysis and the estimation problem, by selecting a critical set of O-D priors that can ensure observability and estimation quality, and considering both the actual values of input parameters and network topology in the observability analysis.

Many studies above have suggested the non-uniqueness of the minimum set of observed links (despite with the same number) for complete link flow observability. The multiplicity of sensor locations may provide flexibility in the selection of links for sensors installation. However, to our best knowledge, there is no explicit uncertainty consideration in the complete link flow observability problem. In this paper, we consider an additional important criterion for selecting sensor locations – the minimum accumulated/propagated uncertainty due to measurement errors of the observed link flows which are unavoidable.

![Fig. 1. Classification of studies on the network sensor location problem (Bianco et al., 2001, 2006).](image-url)
1.3. Main contribution of this paper

In this paper, we develop a robust sensors location model for complete link flow observability in a network with no existing measurement information. The robust optimization is motivated by the fact that the relationship between the unobserved and observed link flows cannot be expressed generally and explicitly as a function of the sensor location scheme (to be elaborated in Remark 1 of Section 2.1). Hence, an explicit objective function expressed in terms of the sensor location scheme cannot be written directly to minimize the accumulated variance of inferred link flows propagated from measurement errors. Instead, this paper proposes an indirect way to resolve this problem. This indirect approach is developed based on the observation that fewer unobserved links involved in the nodal flow conservation equations will have a lower chance of accumulating measurement errors, and hence a lower uncertainty in the inferred link flows.

Mathematically, we formulate this problem as min–max\(^1\) and min–sum binary integer linear programs (BILPs). The features of the proposed robust model are as follows. (1) The feasible domain decomposes the location searching space from the whole network link set into node-specific mutually exclusive subsets. It determines the minimum number of observed links required for complete link flow observability, while avoiding the need of enumerating the link–path incidence matrix or generating the set of linearly independent paths. (2) The objective function minimizes the largest or the cumulative number of unobserved links connected to each node. This treatment reduces the likelihood of incurring secondary inferences or ‘borrowing’ nodal flow conservation information from other nodes, as will be explained later. In addition, information on route choice behaviors and traffic assignment is not needed. Computationally, the BILPs of the proposed models can be solved by a number of existing efficient algorithms for globally optimal solution. Furthermore, we extend the robust models to consider the redundant sensor location problem beyond the complete link observability for uncertainty reduction. The proposed models could be used at the strategic planning stage of sensors installation, where information associated with observed link flows and their measurement errors are not yet available.

The remainder of this paper is organized as follows. Section 2 presents the link flow inference and uncertainty accumulation. Section 3 demonstrates the necessity of considering the accumulated uncertainty as an additional criterion in the NSLP. Section 4 provides the robust location models for complete link observability. Section 5 extends to a redundant location model for uncertainty reduction. Four numerical examples are presented in Section 6 to demonstrate the features and applicability of the proposed models. Finally, concluding remarks are summarized in Section 7.

2. Link flow inference and uncertainty accumulation

2.1. Link flow inference

Link flow inference is to infer the unobserved link flows based on the observed link flows. There are two main approaches to inferring link flows: node-based approach and link-based approach. Specifically, the node-based approach uses the non-centroid node–link flow conservation equations while the link-based approach uses the link–path flow conservation equations. To avoid path enumeration, we adopt the node-based approach suggested by Ng (2012). The nodal flow conservation equation can be written as follows:

\[
(A_U, A_O) \begin{bmatrix} v_U \\ v_O \end{bmatrix} = 0, \quad \text{or} \quad A_U v_U + A_O v_O = 0, \quad (1)
\]

where \(A\) is the non-centroid node–link incidence matrix with the elements defined in Eq. (2); \(A_U\) and \(A_O\) are the submatrices of \(A\) corresponding to the unobserved (denoted by \(U\)) and observed (denoted by \(O\)) links, respectively; \(v_U\) and \(v_O\) are the corresponding unobserved and observed link flow vectors:

\[
A_{ij} = \begin{cases} +1, & \text{if } j \in I(i) \\ -1, & \text{if } j \in O(i) \\ 0, & \text{otherwise} \end{cases}, \quad (2)
\]

where \(i\) is a non-centroid node\(^2\) and \(j\) is a link; \(I(i)\) and \(O(i)\) are the sets of incoming and outgoing links at node \(i\). The number of rows and columns of \(A\) matrix are respectively the number of non-centroid nodes and the number of links (denoted by \(n\) and \(m\)). With \(n\) independent equations \(Av = 0\), we are only able to uniquely determine \(n\) unknown variables of \(v\) while the remainder \(m-n\) variables should be known. In other words, we need to have \(m-n\) observed links for uniquely inferring the \(n\) unobserved links.

To infer the unobserved link flows \(v_U\), the non-centroid node–link incidence matrix corresponding to the unobserved links \((A_U)\) should be invertible. In other words, \(A_U\) should be a square matrix with the dimension of \(n\) (i.e., the number of unobserved links equals the number of non-centroid nodes), and also all columns in \(A_U\) should be linearly independent. Ng (2012) has suggested that \(A_U\) is not necessarily unique.

---

1 The min–max optimization is a dominating paradigm in robust optimization. It primarily deals with non-probabilistic models of robustness with the worst-case orientation.

2 Centroid nodes are nodes where trips are generated from and attracted to, and non-centroid nodes are all other nodes in the network.
Consider with 5),\(\sigma_a\) = \(\lambda_\text{obs}^a\) = \(\lambda_\text{unobs}^a\) = \(\lambda_\text{error}^a\).

More specifically, the unobserved flow on link \(a\) can be inferred from the linear combination of the observed link flows. 

\[
v_a = \sum_{b \in O} \lambda_\text{obs}^b v_b, \quad \forall a \in U, \tag{4}
\]

where \(\lambda_\text{obs}^b\) is the coefficient associated with link \(b\) to infer link \(a\).

**Remark 1.** Let \(\mathbf{x}\) represent the network sensor location scheme, where \(x_a = 1\) if link \(a\) is selected as an unobserved link and \(x_a = 0\) if link \(a\) is selected as an observed link. The specific value of \(\lambda_\text{obs}^a\) in Eq. (4) can be obtained from \(-A_\text{u}^{-1}A_\text{d}\) in Eq. (3). However, the relationship between the unobserved and observed link flows cannot be expressed *generally and explicitly* as a function of the sensor location scheme (i.e., an explicit function of \(-A_\text{u}^{-1}A_\text{d}\) in terms of \(\mathbf{x}\)). The reason is that we rely on the specific value of \(\mathbf{x}\) to partition matrix \(A\) into \(A_\text{u}\) and \(A_\text{d}\) and then to inverse \(A_\text{u}\).

Consider a simple network shown in Fig. 2. This network has one origin (i.e., node 1), two destinations (i.e., nodes 4 and 5), and seven links. Table 1 illustrates the above link flow inference process under two different location schemes \(\mathbf{x}\). In summary, there is no explicit and general expression for \(\lambda_\text{obs}^a\) as a function of the network sensor location scheme \(\mathbf{x}\).

### 2.2. Measurement uncertainty accumulation

In reality, hardware breakdown or communication error inevitably makes the observed link flows subject to measurement error/inaccuracy. In the link flow inference process, the measurement error will be accumulated and propagated to the uncertainty of unobserved link flows. For a given feasible set of observed links that satisfy complete link observability, the unobserved link flows can be uniquely inferred from the observed link flows. Hence, the unobserved link flow estimates themselves do not have the multiplicity issue. With Eq. (4) and the independence assumption of measurement errors, the flow variance on unobserved link \(a\) can be expressed as:

\[
\sigma_a^2(\mathbf{x}) = \sum_{b \in O} (\lambda_\text{obs}^b)^2 \sigma_b^2, \quad \forall a \in U, \tag{5}
\]
Table 2

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Link counters</th>
<th>Equations and accumulated variance of inferred link flows</th>
<th># of unobserved links at node</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equations</td>
<td>Accumulated variance</td>
<td>Node 2</td>
</tr>
<tr>
<td>1</td>
<td>3 4 5 6 7 v1 = v3 + v4 + v5</td>
<td>v1 = v3 + v4 + v5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>3 3 4 5 7 v1 = v3 + v4 + v5</td>
<td>v2 = v3 + v4 + v5</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>2 3 4 5 6 6 v1 = v3 + v4 + v5</td>
<td>v2 = v3 + v4 + v5</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>1 4 5 6 7 v3 = v1 - v4 - v5</td>
<td>v2 = v3 + v4 + v5</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>1 2 4 5 7 v3 = v1 - v4 - v5</td>
<td>v2 = v3 + v4 + v5</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>1 2 4 5 6</td>
<td>v1 = v1 - v4 - v5</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>1 3 5 6 7 v4 = v1 - v3 - v5</td>
<td>v2 = v4 + v5</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>1 2 3 5 7</td>
<td>v4 = v1 - v3 - v5</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>1 2 3 5 6</td>
<td>v4 = v1 - v3 - v5</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>1 3 4 6 7 v3 = v1 - v3 - v4</td>
<td>v4 = v3 + v5</td>
<td>6</td>
</tr>
<tr>
<td>11</td>
<td>1 2 3 4 7</td>
<td>v3 = v1 - v3 - v4</td>
<td>6</td>
</tr>
<tr>
<td>12</td>
<td>1 2 3 4 6</td>
<td>v3 = v1 - v3 - v4</td>
<td>6</td>
</tr>
<tr>
<td>13</td>
<td>2 4 5 6 7</td>
<td>v1 = v5 + v7 - v2</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>1 2 5 6 7</td>
<td>v1 = v5 + v7 - v2</td>
<td>8</td>
</tr>
<tr>
<td>15</td>
<td>1 2 4 6 7</td>
<td>v1 = v5 + v7 - v2</td>
<td>8</td>
</tr>
</tbody>
</table>

where $\sigma_b^2$ is the flow measurement variance of observed link $b$. The accumulated variance of all inferred link flows propagated from measurement errors on all observed link flows is as follows:

$$\sum_{a \in U} \sigma_a^2(x) = \sum_{a \in U} \sum_{b \in O} (\lambda_b)^2 \sigma_b^2.$$  \hspace{1cm} (6)

Recall in Remark 1 that there is no explicit and general formulation of $\lambda_b^2$ as a function of $x$. Hence, $\sigma_b^2$ also does not have an expression as a function of $x$ in Eq. (5). This implies that the sensor locations cannot be directly determined by minimizing the accumulated variance of all inferred link flows (i.e., Eq. (6)). One may use evolutionary algorithms to solve this problem iteratively or compare the accumulated variance after enumerating all feasible location schemes. In order to evaluate the accumulated variance, we still need to perform the link flow inference process (as shown in Eq. (3)) for each location scheme, which involves matrix partition, inversion, and multiplication operations. Depending on the number of possible enumerated location schemes and the number of iterations in evolutionary algorithms, the evaluation efforts could be computationally expensive. Section 3 will give an example to motivate an indirect way to tackle the above difficulty.

3. Motivating example and observation

We further consider the example in Section 2.1. To infer the unobserved link flows, we make use of the flow conservations at the two non-centroid nodes:

Node 2 : \[ v_1 = v_3 + v_4 + v_5, \] \hspace{1cm} (7)

Node 3 : \[ v_2 + v_3 = v_6 + v_7. \] \hspace{1cm} (8)

There are only two independent equations in this simple setting, corresponding to the two non-centroid nodes. Table 2 shows a total of 15 feasible combinations of sensor locations for complete link flow inference. Each location scheme has two unobserved links. All 15 schemes are able to uniquely infer the entire set of unobserved link flows. The reason is that in any of the 15 location schemes with five observed links, the two independent equations can uniquely determine the other two unobserved link flows. However, these location schemes have different inference processes/equations for the unobserved link flows as shown in Table 2.

In general, link measurements are inevitably subject to error/inaccuracy. This error/inaccuracy will be accumulated and propagated to the relevant inferred link flows. The final exhibition is that these feasible location schemes may have quite different inaccuracy accumulation and propagation results. For demonstration purposes, we assume all five independent counters under each location scheme have the same measurement error (e.g., 1 unit). According to the respective inference equations, we can calculate the total accumulated variance of inferred link flows as shown in Table 2. To enhance the reliability of link flow inference, it seems necessary to find a network sensor location scheme with the minimized accumulated uncertainty.
On the other hand, it is not trivial to enumerate and evaluate all possible combinations for large-size realistic networks. Sections 6.2 and 6.4 will show the large number of feasible location schemes that all satisfy complete link flow observability in the Sioux Falls network and Irvine network. In order to evaluate the accumulated variance, we still need to perform the link flow inference in Eq. (3) for each combination, which involves matrix partition, inversion, and multiplication operations. This adds complexity to the network sensor location problem with an explicit consideration of measurement error propagation. Below we provide an observation that will be used to tackle the above difficulty in an indirect manner.

Observation: The link flow inference is conducted based on the flow conservation in Eq. (1) for all non-centroid nodes. Each flow conservation equation is only able to independently determine a single unobserved link flow associated with this node. However, the minimum set of observed links for complete link flow observability may not be able to satisfy this requirement. As shown in schemes 4, 5, 6, 13, 14 and 15, there are two unobserved links associated with a non-centroid node. This means that we have to ‘borrow’ another nodal conservation equation when inferring some unobserved links. In other words, the inference of some unobserved link flows involves a secondary step, which accumulates the measurement errors to the unobserved link flows as shown in Table 2. In summary, the fewer unobserved links involved in the link flow inference equations, the lower chance of accumulating measurement errors, and the lower uncertainty in the inferred link flows. With this observation, instead of directly minimizing the accumulated variance of the inferred link flows, we minimize the largest or the cumulative number of unobserved links connected to each non-centroid node. This treatment could potentially reduce the chance of incurring a secondary inference process or borrowing nodal flow conservation information from other nodes.

4. Robust network sensor location model

In this section, we provide a robust network sensor location model for complete link flow inference while considering the propagation of measurement errors.

4.1. Node-based new links

Consider the node–link incidence matrix with a given ranking of nodes. We define the new links incident to a node as those links not connected to the previous nodes in the ranking (i.e., not included in the previous node rows). Since the minimum set of observed links and the maximum set of unobserved links for complete link flow inference are mutually complementary, we examine how to build the maximum set of unobserved links for complete link flow inference.

Proposition 1. The maximum set of unobserved links to be inferred from other observed links can be found by selecting any single new link connected to each non-centroid node.

Proof. We look at the non-centroid node–link incidence matrix corresponding to the unobserved links ($A_U$).

(a) Since each selected link (i.e., unobserved link) is a new link, not in the previous node rows, the column vector associated with this selected link is linearly independent of the previous vectors. Hence, all columns in $A_U$ are linearly independent.

(b) Also, only a single new link is selected for each non-centroid node (i.e., each row). The number of selected links (i.e., columns) is equal to the number of non-centroid nodes (i.e., rows). Hence, $A_U$ is a square matrix with the dimension equal to the number of non-centroid nodes.

From (a) and (b), $A_U$ is invertible. Accordingly, Eq. (3) can be used to infer all the unobserved link flows via various linear combinations of the observed link flows. Therefore, the maximum set of unobserved links to be inferred from other observed links can be found by selecting any single new link connected to each non-centroid node. This completes the proof. □

Remark 2. Castillo et al. (2014) used the concept of new links to generate the subset of linearly independent paths. In this paper, the introduction of node-based new links can always guarantee the complete link flow observability while avoiding the use of link–path incidence matrix. Essentially, the whole link set is decomposed into multiple mutually exclusive subsets and each subset is connected to a non-centroid node. Each link appears only once in one of the node-specific subsets. There are different ways to construct the set of new links. We could rank the nodes according to their relative importance (e.g., Ivanchev et al., 2016), spatial location, or some particular purposes. However, the exact traffic volume is not needed. The role of constructing new links for each non-centroid node is similar to that of specifying the candidate link set for capacity enhancement in the network design problem without uncertainty (Yang and Bell, 1998) and with uncertainty (Chen et al., 2011). We should point out that the set of new links may not be unique, depending on the ranking of non-centroid nodes.

Now a parallel highway network shown in Fig. 3 is used to demonstrate the concept of new links. This network has four centroid nodes (1, 2, 8, 9), five non-centroid nodes (3, 4, 5, 6, 7), and 14 directed links. Each non-centroid node is connected by four links. This network has also been used by Hu et al. (2009) and Ng (2012). The new links are shown in Table 3. Then, we can construct the maximum set of unobserved links to be inferred from other observed links (and accordingly the minimum set of observed links for complete link observability) by selecting any single new link for each non-centroid node.
One can see that, even for this simple example, there are $4 \times 4 \times 2 \times 2 = 128$ feasible combinations. One can image that the number of feasible combinations will be increasing dramatically for large-scale networks.

4.2. Solution domain

Let $N$ denote the set of non-centroid nodes with the cardinality of $|N|$, and $A_i$ denote the set of new links connected to non-centroid node $i$. We can define the solution domain described by Proposition 1 as follows.

(a) Binary variable

$$x_a = \{0, 1\}, \quad \forall a \in A.$$  \hspace{1cm} (9)

where $x_a = 1$ if link $a$ is selected as an unobserved link, and 0 otherwise.

(b) Only one new link is selected from each non-centroid node

$$\sum_{a \in A_i} x_a = 1, \quad \forall i \in N.$$  \hspace{1cm} (10)

The set of new links connected to node $i$ has the following relationship:

$$A = \bigcup_{i \in N} A_i,$$

$$A_i \bigcap A_{N/i} = \emptyset.$$  \hspace{1cm} (11)

In other words, the new links connected to each non-centroid node come from the multiple mutually exclusive subsets and their union constitutes the whole link set. Hence, Eq. (10) implicitly ensures that the number of selected links (i.e., unobserved links) is always equal to the number of non-centroid nodes as shown below:

$$\sum_{a \in A} x_a = \sum_{i \in N} \left(\sum_{a \in A_i} x_a\right) = \sum_{i \in N} 1 = n.$$  \hspace{1cm} (12)

In summary, all feasible solutions that satisfy Eqs. (9) and (10) can generate the minimum set of observed links for complete link flow inference and the maximum set of unobserved links to be inferred from other observed links. The number of solution variables is equal to the number of directed links and the number of equality constraints is equal to the number of non-centroid nodes. The above solution domain is represented by the node–link incidence matrix. It avoids the need to have a link–path incidence matrix or to know the used paths a priori. Also, the dimension of node–link incidence matrix is much smaller than that of the link–path incidence matrix.

4.3. Objective function

Following the observation in Section 3, we first tackle the modeling difficulty with a min–max problem – a leading paradigm in robust optimization. The objective is to minimize the largest number of unobserved links connected to each
non-centroid node as follows:
\[
\min\ \max_{x_i} Z_i(x) = \sum_{a \in A} x_a \gamma_a^i,
\]
where \(i\) is a non-centroid node; \(\gamma_a^i\) is the node–link indicator: \(\gamma_a^i = 1\) if link \(a\) is connected to node \(i\) and 0 otherwise; \(\sum_{a \in A} x_a \gamma_a^i\) is the number of unobserved links connected to a non-centroid node \(i\) (recall that \(x_a = 1\) if link \(a\) is an unobserved link). This objective function could potentially reduce the chance of incurring a secondary inference or borrowing nodal flow conservation information from other nodes.

### 4.3.1. Model 1: min–max problem

For completeness, we present the proposed min–max robust network sensor location model for complete link flow observability as follows:
\[
\min\ \max_{x_i} Z_i(x) = \sum_{a \in A} x_a \gamma_a^i,
\]
\[
\text{s.t. } \sum_{a \in A} x_a = 1, \ \forall i \in N,
\]
\[
x_a = \{0, 1\}, \ \forall a \in A.
\]

The feasible set guarantees that the selected (unobserved) links correspond to the minimum set of observed links (or the maximum set of unobserved links) for complete link flow observability; the objective function is to minimize the largest number of unobserved links connected to each non-centroid node. The inputs of this model are: (a) set of new links (i.e., \(A_i\)), and (b) node–link incidence matrix (i.e., \(\gamma_a^i\)).

The above formulation is a min–max problem, which may not be efficiently solvable. Below we reformulate it as an equivalent minimization problem by adding just one extra variable \(y\) and a set of linear inequality constraints:
\[
\min_{x, y} y,
\]
\[
\text{s.t. } \sum_{a \in A} x_a \gamma_a^i \leq y, \ \forall i \in N,
\]
\[
\sum_{a \in A} x_a = 1, \ \forall i \in N,
\]
\[
x_a = \{0, 1\}, \ \forall a \in A.
\]

One can see that the above reformulation is a mixed integer linear programming (MILP). A number of existing efficient algorithms in commercial software packages could be used for its globally optimal solution. For completeness, the equivalency is provided below.

**Proposition 2.** The solution to Eqs. (17)–(20) is also a solution to Eqs. (14)–(16).

**Proof.** Note that Eq. (18) is valid for all non-centroid nodes \(\forall i \in N\). Hence, we have \(y \geq \max_{i} \sum_{a \in A} x_a \gamma_a^i\). For a given location scheme \(x = \{x_a\}\), \(\max_{i} \sum_{a \in A} x_a \gamma_a^i\) defines the lower bound of the feasible range of \(y\). Accordingly, the minimized \(y\) among all feasible location schemes (i.e., Eq. (17)) is equal to the minimum \(\max_{i} \sum_{a \in A} x_a \gamma_a^i\) among all feasible location schemes (i.e., Eq. (14)). This completes the proof. \(\square\)

### 4.3.2. Model 2: min–sum problem

The min–max model explicitly considers the worst case, i.e., the largest number of unobserved links connected to each non-centroid node. However, the nodal percentage with this worst case could be quite low in some cases. To further consider the distribution of the number of unobserved links at each non-centroid node, we provide an alternative formulation with the minimization of the cumulative number of unobserved links at all non-centroid nodes.
\[
\min_{x} \sum_{i \in N} \sum_{a \in A} x_a \gamma_a^i,
\]
\[
\text{s.t. } \sum_{a \in A} x_a = 1, \ \forall i \in N,
\]
\[
x_a = \{0, 1\}, \ \forall a \in A.
\]
Note that Eq. (21) includes the multiple occurrences of some unobserved links in the inference equations. This consideration corresponds to the uncertainty accumulation. For example, in Case 4 of Table 2, node 2 has one unobserved link (i.e., link 3) and node 3 has two unobserved links (i.e., links 3 and 2). The inference of link 2 inherits the uncertainty of inferring link 3. The cumulative number of unobserved links is 3, which is also the number of times of link flow inference (i.e., 1 time for link 3 and 2 times for link 2). Mathematically, this model is a binary integer linear programming (BILP). The linear programming nature of the above robust location models may make the optimal solution non-unique. However, the optimal solution is global in terms of the objective value. From this viewpoint, the robust location models could also be considered as a benchmark for comparison. In other words, without obtaining the optimal objective values by solving the optimization models, we are not aware of the quality (in terms of uncertainty accumulation performance) of sensor location trials.

Remark 3. To better explain the comparison with existing methods, we ‘divide’ our proposed models into two sequential parts: (a) preparing the model input, particularly the set of new links, and (b) solving the robust location models. The purpose of part (a) together with the feasible solution set is the same as the existing methods on the complete link flow observability problem (i.e., all feasible solutions correspond to the minimum set of observed links for complete link flow observability). From this perspective, our approach provides a set of solutions for further uncertainty consideration, instead of a single solution resulted from existing methods without uncertainty consideration. Part (b) involves solving the proposed models, BILP or MILP, efficiently by making use of many existing algorithms in commercial software packages. Due to this computational tractability, we only need a small incremental computation effort to achieve this uncertainty consideration benefit in determining the optimal sensor location for complete link flow observability.

4.4. Consideration of non-uniform measurement errors

As mentioned in Zhou and List (2010), although the actual values of sensor measurements are unknown before installing the sensors, the magnitude of measurement errors can still be estimated via an analogue from the same type of sensors at similar locations or related studies in other areas. Relative to the uniform measurement variance, the inclusion of non-uniform measurement variances can further refine and differentiate various optimal location schemes, all of which can ensure the complete link flow observability but with different accumulated uncertainties (i.e., objective function values in our context).

Now suppose the measurement errors of all links are available and independent. Let \( \sigma^2_{a,\text{min}} \) and \( \sigma^2_{a,\text{max}} \) denote the minimum and maximum variance among all links, respectively. Below we modify the proposed mix–max and min–sum models to account for the link-specific non-uniform variance of measurement errors across links. Since the constraint set keeps intact, we only present the modified objective functions as follows:

\[
\min_x \max_i Z_i(x) = \sum_{a \in A} x_a \cdot \gamma^i_a \cdot \sigma^2_a, \quad (24)
\]

\[
\min_x \sum_{i \in N} \sum_{a \in A} x_a \cdot \gamma^i_a \cdot \sigma^2_a. \quad (25)
\]

Since the model structure and property have not been changed, the modified models are still readily solvable. Note that the solution non-uniqueness of each model makes the comparison of location schemes difficult and may not be meaningful. Instead, we examine the model relationship by comparing the optimal accumulated uncertainty (i.e., objective function values) between the uniform-variance case and the non-uniform-variance case.

Since \( \sigma^2_{a,\text{min}} \leq \sigma^2_a \leq \sigma^2_{a,\text{max}}, \) \( \forall a, \) we have

\[
\sigma^2_{a,\text{min}} \sum_{a \in A} x_a \cdot \gamma^i_a \leq \sum_{a \in A} x_a \cdot \gamma^i_a \cdot \sigma^2_a \leq \sigma^2_{a,\text{max}} \sum_{a \in A} x_a \cdot \gamma^i_a, \quad \forall i, \quad (26)
\]

\[
\sigma^2_{a,\text{min}} \sum_{i \in N} \sum_{a \in A} x_a \cdot \gamma^i_a \leq \sum_{i \in N} \sum_{a \in A} x_a \cdot \gamma^i_a \cdot \sigma^2_a \leq \sigma^2_{a,\text{max}} \sum_{i \in N} \sum_{a \in A} x_a \cdot \gamma^i_a. \quad (27)
\]

The accumulative uncertainty has the following relationship for the min–max and min–sum models:

\[
\sigma^2_{a,\text{min}} \min_x \max_i Z_i(x) \leq \min_x \max_i \sum_{a \in A} x_a \cdot \gamma^i_a \cdot \sigma^2_a \leq \sigma^2_{a,\text{max}} \min_x \max_i Z_i(x). \quad (28)
\]

\[
\sigma^2_{a,\text{min}} \cdot \min_x \max_i \sum_{a \in A} x_a \cdot \gamma^i_a \leq \min_x \sum_{i \in N} \sum_{a \in A} x_a \cdot \gamma^i_a \cdot \sigma^2_a \leq \sigma^2_{a,\text{max}} \cdot \min_x \sum_{i \in N} \sum_{a \in A} x_a \cdot \gamma^i_a. \quad (29)
\]

The optimal accumulated uncertainty with non-uniform measurement variance is always bounded by the optimal accumulated uncertainty with uniform measurement variance, as scaled by the minimum link measurement variance and the maximum link measurement variance, respectively. The optimal accumulated uncertainty between the non-uniform-variance and uniform-variance cases gets closer with the decrease of measurement error deviation. In other words, the assumption of uniform measurement variances actually forms an upper bound accumulated uncertainty for the network, or a worst case scenario.
5. Extension: uncertainty reduction-oriented redundant sensor location problem

An extended question closely related to the robust complete link observability is: How many redundant sensors are needed for a certain level of uncertainty reduction? To address this question, we modify the min-sum or min-max model as follows:

\[
\min_x \sum_{i \in N} \left( \sum_{a \in A_i} \sum_{a \in A} x_a \cdot x_a y_a \right), \quad \text{or} \quad \min_x \max_i \left( \sum_{a \in A_i} \sum_{a \in A} x_a y_a \right),
\]

\[
\max_x \sum_{a \in A} x_a,
\]

s.t. \( \sum_{a \in A_i} x_a \leq 1, \forall i \in N, \)

\[
x_a = \{0, 1\}, \forall a \in A.
\]

Recall that \( x_a = 1 \) if link \( a \) is an unobserved link. Compared to Eqs. (14)–(16) or Eqs. (21)–(23), the above model has two main differences: (a) Eq. (31) is to minimize the sensors installation and operating cost by maximizing (or minimizing) the total number of unobserved (or observed) links. The two objectives consider a tradeoff between the number of observed (or unobserved) links and the accumulated uncertainty (indirectly via the times of link flow inference). Installing sensors on all links is a feasible solution, but it has the worst objective value in Eq. (31) (i.e., zero). (b) Eq. (32) is to relax the number of unobserved links at non-centroid node \( i \). This is where the redundancy comes from. Eqs. (32) and (33) imply that \( \sum_{a \in A} x_a = 0 \) or 1.

- If \( \sum_{a \in A} x_a = 1 \) (like in Section 4), we allow only one link to be selected from each non-centroid node. Together with the mutually exclusive subsets of new links at each non-centroid node, we always select the minimum number of unobserved links, which equals the number of non-centroid nodes as shown in Eq. (12). When \( \sum_{a \in A} x_a = 1 \) for all non-centroid nodes, the objective function in Eq. (30) is exactly the same as the previous min-max and min-sum models.

- If \( \sum_{a \in A} x_a = 0 \), all new links connected to node \( i \) have been installed with sensors, and there is no flow inference at this node (no new link needs to be inferred from this node). We could refer to these nodes as ‘ineffective nodes’, since their nodal flow conservation equations are not effectively used in the link flow inference. However, it does not mean that \( \sum_{a \in A} x_a y_a = 0 \) due to network connection. Hence, it will be biased to directly minimize \( \sum_{i \in N} \sum_{a \in A} x_a y_a \). Instead, we use Eq. (30) to avoid the over-consideration. Specifically, if some nodes have \( \sum_{a \in A} x_a = 0 \), they are not considered in the accumulated uncertainty. In essence, the objective function in Eq. (30) is the largest number of unobserved links at each effective node and the cumulative number of unobserved links at all effective nodes.

**Remark 4.** A similar ‘correction’ to the case of over consideration will also be used in Section 6.2, where there is no centroid node specification and we simply specify node 24 as an ineffective node. The flow conservation equation at node 24 is not used to infer the twenty-three unobserved links. Accordingly, we did not include node 24 in Table 6. For a network with specialized centroid nodes, all non-centroid nodes are effective nodes since their nodal flow conservation equations have to be used for the complete link flow observability. In contrast, Eq. (30) endogenously determines the ineffective nodes (i.e., \( \sum_{a \in A} x_a = 0 \)) by considering the effect of uncertainty reduction.

The model in Eqs. (30)–(33) has a bi-objective structure. To make it solvable, we can use the \( \varepsilon \)-constraint method (Miettinen, 1999) to transform it into a single-objective optimization problem. For example,

\[
\min_x \sum_{i \in N} \left( \sum_{a \in A_i} \sum_{a \in A} x_a \cdot x_a y_a \right), \quad \text{or} \quad \min_x \max_i \left( \sum_{a \in A_i} \sum_{a \in A} x_a y_a \right),
\]

s.t. \( \sum_{a \in A} x_a \geq n - c, \)

Eqs. (32) – (33),

where \( n \) is the number of non-centroid nodes, i.e., the maximum number of unobserved links for complete link flow observability; \( c \) is the additional number of observed links beyond complete link flow observability. When Eq. (35) becomes \( \sum_{a \in A} x_a \geq n \) (i.e., \( c = 0 \)), together with \( \sum_{a \in A} x_a \leq 1 \) in Eq. (32), we always have \( \sum_{a \in A} x_a = 1 \). This will degenerate to the robust model as presented in Section 4.3. By varying the value of \( c \), the formulation could be used to evaluate the marginal benefit of installing an additional sensor (beyond complete link observability) on the uncertainty reduction. Note that the objective function in Eq. (34) is quadratic, rendering the redundant sensor location model as a binary integer quadratic programming.

Alternatively, we can also treat the implicit uncertainty consideration as a constraint.

\[
\max_x \sum_{a \in A} x_a,
\]
where $c$ is a desired uncertainty level (e.g., 10% reduction relative to the robust location problem). This formulation could answer the question of ‘At least how many additional sensors are needed for a certain degree of uncertainty reduction’. Note that the above redundant models can also cater for some specialized analysis purposes. For example, analysts may want to enhance the estimation quality of link flows at some particular area rather than the whole network, by installing redundant sensors. In this case, we can readily modify the objective function to confine nodes only within the area of interest.

6. Numerical examples

6.1. Example 1: parallel highway network

We continue to use the parallel highway network shown in Fig. 3 and the set of new links tabulated in Table 3 to demonstrate the features of the proposed robust sensor location models. The solution of the MILP problem in Eqs. (17)–(20) is shown in Fig. 4, which is the same as the solution to the BILP in Eqs. (21)–(23). Nine observed links are required to infer the other five unobserved links. The largest number of unobserved links is 2 at node 6, which correspond to link 9 and link 11.

Now we examine the inference process of the unobserved link flows from the observed link flows. In general, we can use Eq. (3) to obtain the inference equations (i.e., via incidence matrix partition, inversion, and multiplication). For this type of illustrative networks, we can also use the nodal flow conservation directly.

\[
\begin{align*}
\text{node 3:} & & v_3 &= v_5 + v_6 - v_1 \\
\text{node 4:} & & v_4 &= v_7 + v_8 - v_2 \\
\text{node 5:} & & v_5 &= v_7 + v_9 - v_{10} \\
\text{node 7:} & & v_{13} &= v_9 + v_{10} - v_{14} \\
\text{node 6:} & & v_{11} &= v_6 + v_9 - v_{12} &= v_6 + v_5 + v_7 - v_{10} - v_{12}.
\end{align*}
\]

where node 6 is connected to two unobserved links (link 9 and link 11) while each of the other four non-centroid nodes only involves a single unobserved link. To infer $v_{11}$, we need to borrow information from node 5 (i.e., the inference of $v_9$). This secondary step will accumulate and propagate the measurement errors of links 5, 7, and 10 to link 11. For example, if all observed link flows have the independent measurement variance of 1 unit, links 3, 4, 9, and 13 will have the accumulated variance of 3 units. However, link 11 has the accumulated variance of 5 units, significantly reducing the reliability of its inferred link flow.

To further explore the uncertainty characteristics in the link flow inference process, we enumerate all 128 feasible combinations and summarize the uncertainty characteristics in Table 4. One can see that these feasible combinations have different numbers of non-centroid nodes with 1, 2, and 3 unobserved links as indicated in the Distribution column in Table 4. There are five types of distributions: (4, 1, 0), (4, 0, 1), (3, 2, 0), (3, 1, 1), and (2, 3, 0). Even within the same type, different location schemes have distinct accumulated variances, e.g., (3, 2, 0) with variances of 19 and 21, and (3, 1, 1) with variances of 19, 21, and 23. All 128 feasible combinations have the same number of observed links (i.e., 9) and also are able to ensure the complete link flow observability. However, such a dispersed uncertainty accumulation performance indeed highlights the importance of considering uncertainty in sensors location planning and the need of a robust optimization model for making a wise location decision.

Now we compare Model 1 (i.e., min–max problem) and Model 2 (i.e., min–sum problem) in Table 4 with the values of \( \max_i \sum_{a \in A} x_a y_a^i \) and \( \sum_{i \in N} \sum_{a \in A} x_a y_a^i \). For this particular example, the min–sum model seems to have a better distinguishing capability compared to the min–max model. For example, \( \max_i \sum_{a \in A} x_a y_a^i = 2 \) corresponds to \( \sum_{i \in N} \sum_{a \in A} x_a y_a^i \), 6, 7, and 8.

![Fig. 4. A robust location of the parallel network for complete link observability.](image-url)
However, the min–max problem still has its own appeal. When \( \sum_{i \in N} \sum_{a \in A} x_{ai} y_{ai} = 7 \), \( \max \sum_{i \in N} \sum_{a \in A} x_{ai} y_{ai} \) could be both 2 and 3. From this viewpoint, the min–max and min–sum criteria could work together to provide a more comprehensive assessment of location schemes from the worst case and the aggregated perspective.

Recall that the measurement errors of the observed link flows are unknown at the planning stage. The above variance analysis could only be considered as a post-analysis. The two location models proposed in Section 4 do not explicitly consider the link flow inference and variance calculation due to the absence of an explicit/general expression of linear combination coefficients as a function of the location scheme \( x \). However, the two models have an implicit consideration of uncertainty accumulation and propagation via the number of unobserved links connected to each non-centroid node. The treatment could potentially reduce the chance of incurring a secondary inference process or borrowing flow conservation information from other nodes. The fewer unobserved links associated with each non-centroid node, the lower chance of accumulating measurement errors in the link flow inferences, and the lower uncertainty in the inferred link flows. As shown in Table 4, when \( \sum_{i \in N} \sum_{a \in A} x_{ai} y_{ai} \) changes from 6 to 8, the accumulated variance also increases from 17 to 23.

6.2. Example 2: Sioux Falls network

In this section, we use the Sioux Falls network, a well-known network used in many transportation network analyses, to demonstrate the applicability and adaptability of the robust location models. This network has 24 nodes and 76 links. We do not specify the centroid nodes in this example. This consideration is particularly justifiable for two applications: (1) Different zonal resolutions: different analyses (e.g., large-scale urban transportation planning versus small-scale sub-area analysis or traffic impact analysis of a road construction project) require different zonal systems. However, traffic monitoring with traffic counts is common information for these applications with different spatial scales. (2) Long-term counting: as suggested by Ng (2012), it is reasonable to argue that centroids do not exist when traffic counts are collected over an entire day. One is typically interested in average traffic counts over a sufficiently long time period (e.g., average daily traffic) to smooth out the variability. Travelers exit the road network at a node (e.g., go to work), and they will re-enter the network (e.g., go home after work) within the same day. Catering to this consideration, the proposed robust location models are still applicable by only removing any one row from the node-link incidence matrix. After deleting one row from the node-link incidence matrix, the new matrix has a full rank (Ahuja et al., 1993). The rank (i.e., number of nodes minus 1) is the maximum number of unobserved links that can be inferred from other observed links.

First of all, we construct the node-based new links as discussed in Section 4.1. From Table 5, there are a total of \( 2^{37} = 137,438,953,472 \) feasible combinations. All of them are able to guarantee the complete link flow observability. However, it is not feasible to derive the link flow inference equations (i.e., Eqs. (3) and (4)) for all these combinations and then compare their accumulated variance (i.e., Eqs. (5) and (6)). This computational difficulty further highlights the need of a robust location optimization model with a prior uncertainty consideration at the planning stage.

Fig. 5 shows the optimal location schemes from the min–max model and min–sum model. Both models generate 23 unobserved links as showed by the dashed lines. In both models, the largest number of unobserved links at each node is 3; the cumulative number of unobserved links (including the multiple occurrences) \( \sum_{i \in N} \sum_{a \in A} x_{ai} y_{ai} \) is 43. Table 6 further shows the number of nodes connected with 1, 2, and 3 unobserved links. One can see that more than 65% (i.e., 15/23 versus 16/23) nodes have more than 1 unobserved link, and accordingly need to borrow other nodal flow conservation equations to infer all their connected unobserved links. The final exhibition is that the optimal location scheme needs 43 times of flow inference in order to achieve the full link observability, which is much larger than 23 nodal flow conservation equations.

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncertainty summary of the 128 feasible combinations.</td>
</tr>
<tr>
<td>Variance</td>
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<tr>
<td>Variancea</td>
</tr>
<tr>
<td>17</td>
</tr>
<tr>
<td>19</td>
</tr>
<tr>
<td>21</td>
</tr>
<tr>
<td>23</td>
</tr>
</tbody>
</table>

a: Variance: total accumulated variance of inferred link flows.
b: #: number of solutions; there are 128 solutions in total, and % in parenthesis is a percentage of 128 solutions.
c: Distribution (3, 1, 1): 3 nodes have one unobserved link, 1 node has two unobserved links, and 1 node has three unobserved links.
d: Max: maximum number of unobserved links at each non-centroid node.
e: Sum: cumulative number of unobserved links (including multiple occurrences).
Table 5
Node-based new links of the Sioux Falls network.

<table>
<thead>
<tr>
<th>Node</th>
<th>Links</th>
<th>New links</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
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<td>6, 7, 8, 35</td>
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<td>4</td>
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<td>9, 10, 11, 31</td>
</tr>
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<td>5</td>
<td>9, 11, 12, 13, 15, 23</td>
<td>12, 13, 15, 23</td>
</tr>
<tr>
<td>6</td>
<td>4, 12, 14, 15, 16, 19</td>
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</tr>
<tr>
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<td>23</td>
<td>42, 70, 71, 72, 73, 76</td>
<td>73, 76</td>
</tr>
</tbody>
</table>

(a) from min-max model   (b) from min-sum model

Fig. 5. Optimal location schemes of the Sioux Falls network.

Table 6
Inference and uncertainty summary of the Sioux Falls network.

<table>
<thead>
<tr>
<th>Model</th>
<th># of nodes with ( t ) unobserved links</th>
<th>Number of times of flow inference</th>
<th>Accumulated variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( t = 1 )</td>
<td>( t = 2 )</td>
<td>( t = 3 )</td>
</tr>
<tr>
<td>Min–max</td>
<td>8</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Min–sum</td>
<td>7</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>
Without loss of generality, we only show the detailed inference processes for the min–max optimal solution in Table A1. The bold links represent the secondary inference and the potential accumulation of measurement errors. Based on the link flow inference equations, we can also conduct a post-variance analysis by assuming that all observed links are independent with the measurement variance of one unit. From Table 6, the min–max and min–sum optimal solutions have the accumulated variance of 205 and 209, respectively. This slight difference could be further explained by Fig. 6. The accumulation of measurement errors can be shown by the number of observed links used to infer each unobserved link as varied from 3 to 17.

6.3. Example 3: redundant sensor location model

Below we continue to use the Sioux Falls network in Example 2 to briefly demonstrate the redundant min–sum model in Eqs. (34) and (35). The parameter $c$ is varied from 0 to 10. After solving the model for each value of $c$, we also conduct a post-variance analysis. However, we cannot directly follow Section 2.1, since $A_U$ is not square anymore (i.e., the number of columns is smaller than the number of rows). We need to delete the rows with $\sum_{a \in A} x_{ai} = 0$ (i.e., ineffective nodes). Fig. 7 shows the optimal objective value at each $c$ value (i.e., the cumulative number of unobserved links at all effective nodes $\sum_{i \in N} (\sum_{a \in A} x_{ai} \cdot \sum_{a \in A} x_{ai} y'_{aj})$) and the corresponding total variance of inferred link flows. One can see that the inference uncertainty is reduced with the installation of additional sensors. It seems that the uncertainty reduction gradually becomes less substantial with the increase of $c$. Fig. 8 further shows the composition of the left panel of Fig. 7. For the worst case consideration, we need at least five additional observed links to reduce $\max_i (\sum_{a \in A} x_{ai} \cdot \sum_{a \in A} x_{ai} y'_{aj})$ from 3 to 2, and 10 links to 1. In general, the redundant min–sum model allocates the additional sensors to reduce the number of effective nodes with more unobserved links.

6.4. Example 4: Irvine network

To further demonstrate the applicability of the proposed robust location models, we conduct a real case study in the City of Irvine, Orange County, California. This network, as shown in Fig. 9, consists of three major freeways and several arterials in the City of Irvine. The network data were extracted from the Orange County Transportation Analysis Model, which contains data for the whole county. The extracted network is composed of 162 nodes, 496 links, 39 traffic analysis zones (TAZs), and 28 external stations. This real network has been used as a test network for the traffic counting location problem (Chootinan et al., 2005a; Zhou and List, 2010), the automatic vehicle identification (AVI) reader location problem (Chen et al., 2004, 2010a; Fei et al., 2007; Zhou and List, 2010), and the path flow/OD estimation problem (Chen et al., 2005, 2009, 2010b; Chootinan et al., 2005a; Chootinan and Chen, 2011). It serves as a real application for testing the robust network sensor location models proposed in this paper.
Since the Irvine network is more complex (much larger in terms of numbers of nodes, links, and OD pairs) than the previous examples, we develop the following procedure to prepare the new link set $A_i$ (i.e., an input of the proposed models). Recall that the new links incident to a node are defined as those links not included in the previous node rows (i.e., not connected to the previous nodes). Essentially we need to have a ranking of nodes. Let $\delta^i_u$ denote the node–new link
incidence indicator: if $\delta^i_a = 1$, link $a$ is a new link incident to node $i$, and $\delta^i_a = 0$, otherwise. Let $N_1$ and $N_2$ denote the set of nodes that have and have not been ranked, $N_1 \cup N_2 = N$ (the whole set of non-centroid nodes).

Based on the above set of node-based new links, there are more than 1.36E72 feasible combinations, all of which could guarantee the complete link flow observability. However, it is not computationally feasible to derive the link flow inference equations (i.e., Eqs. (3) and (4)) for all the combinations and then compare their accumulated variance (i.e., Eqs. (5) and (6)). This difficulty further necessitates a location optimization model with a prior uncertainty consideration. The optimization results are presented in Table 7. In the min–max model, the largest number of unobserved links at each node is 3, while it is 4 in the min–sum model. In the two models, more than 60% and 46% nodes have more than 1 unobserved link, and thus both need to borrow other nodal flow conservation equations to infer all their connected unobserved links. Accordingly, their optimal location schemes need 280 and 259 times of flow inference for achieving the full link observability, which is much larger than 162 nodal flow conservation equations. For these two optimal location schemes, we also conduct a post-variance analysis by assuming that all observed links are independent with the measurement variance of one unit. Their accumulated variances are 1067 and 966. For this particular case, the min–sum model seems to perform slightly better than

### Table 7
Inference and uncertainty summary of the Irvine network.

<table>
<thead>
<tr>
<th>Model</th>
<th># of nodes with $t$ unobserved links</th>
<th># of times of flow inference</th>
<th>Accumulated variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t=1$</td>
<td>$t=2$</td>
<td>$t=3$</td>
</tr>
<tr>
<td>Min–max</td>
<td>64</td>
<td>78</td>
<td>20</td>
</tr>
<tr>
<td>Min–sum</td>
<td>86</td>
<td>56</td>
<td>19</td>
</tr>
</tbody>
</table>

Fig. 9. Irvine network.
the min–max model. In summary, the robust location optimization models are applicable in realistic networks for a prior uncertainty consideration at the planning stage.

7. Concluding remarks

A few elegant methodologies have been developed to identify the minimum set of observed links for the complete link observability problem. However, this set has a high chance to be non-unique. Various combinations with the same minimum number of links could give unique inferred link flows but with different inference processes. In this paper, we proposed an additional important criterion to identify this minimum set – the minimum accumulated uncertainty of the link flow inference. The observed link flows are inevitably subject to measurement errors. Instead of just staying localized, the measurement errors are accumulated and propagated to the uncertainty of unobserved link flows, reducing the reliability of link flow inference.

Methodologically, we developed robust network sensor location models for complete link flow observability, while indirectly considering the propagation of measurement errors in the link flow inference. Our two observations underpinned the model development: (1) the relationship between the unobserved and observed link flows cannot be expressed generally and explicitly as a function of the location scheme. This prevents from the direct minimization of the accumulated variance of inferred link flows. (2) For a node connected with more than one unobserved link, we need to borrow other nodal flow conservation equations as a secondary/middle step in the full inference process. The fewer unobserved links involved in the nodal flow conservation equations, the lower chance of accumulating measurement errors, and the lower uncertainty in the inferred link flows. This motivates an indirect way to capture the uncertainty propagation.

Mathematically, we formulated the problem as min–max and min–sum binary integer linear programs with different focuses in the objective functions (i.e., the worst case versus the aggregated performance). The min–max model minimizes the largest number of unobserved links connected to each non-centroid node while the min–sum model minimizes the cumulative number of unobserved links connected to all non-centroid nodes. The objective functions have potential to reduce the chance of incurring the secondary inference or borrowing flow conservations from other nodes. In both models, the feasible domain uses the concept of node-based new links to decompose the whole network link set into node-specific mutually exclusive subsets. This decomposition guarantees complete link flow observability while avoiding the use of link–path incidence matrix. Computationally, the resultant binary integer linear program permits many existing efficient algorithms for the globally optimal solution. Furthermore, we extended the robust models to consider the redundant sensor location problem beyond the complete link flow observability for uncertainty reduction. Different network experiments demonstrated the features and applicability of the proposed robust and redundant sensor location models for complete link flow observability. The proposed models are useful at the strategic planning stage of sensors installation in a new network/region, where information associated with observed link flows and their measurement errors are not yet available. As a future research, it would be valuable to consider both the non-uniform measurement variances and dependency across links in the link flow observability problem.

Acknowledgments

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Appendix

Table A1.

<table>
<thead>
<tr>
<th>Node</th>
<th>Link flow inference equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_3 = v_1 + v_2 - v_5$</td>
</tr>
<tr>
<td>2</td>
<td>$v_4 = v_1 + v_4 - v_3 = v_2 + v_4 - v_1 - v_2 + v_5 = v_1 + v_2 - v_5$</td>
</tr>
<tr>
<td>3</td>
<td>$v_{35} = v_5 + v_6 + v_2 - v_3$</td>
</tr>
<tr>
<td>4</td>
<td>$v_{31} = v_8 + v_9 + v_{40} - v_9 - v_{11}$</td>
</tr>
</tbody>
</table>

(continued on next page)
Table A1 (continued)

<table>
<thead>
<tr>
<th>Node</th>
<th>Link flow inference equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( v_{16} = v_4 + v_{12} + v_{19} - v_9 - v_{23} )</td>
</tr>
<tr>
<td>6</td>
<td>( v_{16} = v_4 + v_{12} + v_{19} - v_{14} - v_{18} = (v_{14} - v_2 + v_3) + v_{12} + v_{19} - v_{14} - v_{18} - (v_{11} + v_{12} + v_{13} - v_9 - v_{23}) = -v_2 + v_5 + v_{19} - v_{11} - v_{13} + v_9 + v_{23} )</td>
</tr>
<tr>
<td>7</td>
<td>( v_9 = v_7 + v_{18} - v_{14} )</td>
</tr>
<tr>
<td>8</td>
<td>( v_{24} = v_9 + v_{26} + v_{12} + v_{17} - v_7 - v_9 + v_{24} = (v_9 + v_{17} + v_{18} - v_{14}) + v_{24} + v_{12} - (v_2 + v_5 + v_{19} - v_{11} - v_{13} + v_9 + v_{23}) = v_{17} - v_{47} )</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{v_9}{v_8} = v_{14} + v_{12} + v_{19} - v_9 - v_{23} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>10</td>
<td>( v_{25} = v_1 + v_{21} + v_{26} - v_{31} - v_{24} + v_{11} + v_{26} - v_{23} - (v_{18} - v_{19} + v_{22} + v_{24} + v_{12} + v_5 + v_{19} - v_{11} + v_{13} - v_9 - v_{23} - v_{47}) )</td>
</tr>
<tr>
<td>11</td>
<td>( v_{26} - v_{18}, \frac{v_{14} + v_{12}}{v_{12}} - v_5 = v_{11} + v_4 + v_7 )</td>
</tr>
<tr>
<td>12</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>13</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>14</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>15</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>16</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>17</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>18</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>19</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>20</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>21</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>22</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
<tr>
<td>23</td>
<td>( v_{19} = v_{25} + v_{14} - v_{11} + v_{12} - v_{13} - v_{27} - v_{30} )</td>
</tr>
</tbody>
</table>

References


