Continuous Separating Families in Ordered Spaces and Strong Base Conditions

by

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Abstract: In this paper we study the role of Stepanova's continuous separating families in the class of linearly ordered and generalized ordered spaces and we construct examples of paracompact spaces that have strong base properties (such as point-countable bases or σ-disjoint bases), have continuous separating families, and yet are non-metrizable.

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1. Introduction.

A space X is called a *Urysohn* space if for each pair of distinct points $x, y \in X$ there is a continuous function $f : X \to \mathbb{R}$ with the property that $f(x) \neq f(y)$. In her papers [S1] and [S2], E.N. Stepanova studied spaces in which the separating functions in a Urysohn space could be chosen in a natural manner. She defined that a space X has a *continuous separating family* if there is a continuous function $\Phi : X^2 - \Delta \to C_u(X)$, where $C_u(X)$ is the space of continuous real-valued functions on X with the uniform convergence topology, such that if $f_{x,y} = \Phi(x, y)$, then $f_{x,y}(x) \neq f_{x,y}(y)$. (The collection $\{f_{x,y} : (x, y) \in X^2 - \Delta\}$ is then called a *continuous separating family* for X.) To have a continuous separating family is a very strong property, as can be seen from Stepanova's theorem

1.1 Theorem: *Suppose X is a paracompact p-space in the sense of Arhangel'skii. Then the following are equivalent:*

a) X is metrizable;

b) X has a G_δ-diagonal;

c) X has a continuous separating family.

Previous research shows that there are many parallels between the metrization theory for compact Hausdorff spaces (and for their p-embedded subspaces) and the metrization theory for linearly ordered topological spaces (and for their p-embedded subspaces). In this paper, we show that a weak form of that parallelism exists if one considers Stepanova's continuous separating families. We show that the existence of a continuous separating family in a generalized ordered spaces does yield hereditary paracompactness, and that in separable generalized ordered spaces, the existence of a continuous separating family does give a G_δ-diagonal. However, for ordered spaces with uncountable cellularity, our examples show that the expected parallelism breaks down. In addition, our examples clarify the interactions between continuous separating families and certain very strong base properties (e.g., σ-disjoint bases or point-countable bases).

In Section 2 of this paper, we explore the role of continuous separating families in linearly ordered and generalized ordered spaces, and in Section 3 we use ordered space constructions to investigate the interaction of continuous separating families with certain strong base properties.

Recall that a *generalized ordered space* (GO-space) is a Hausdorff space $(X, \mathcal{T})$ equipped with a linear order $<$ such that the topology $\mathcal{T}$ has a base consisting of convex sets. It is known that GO-spaces are precisely the topological subspaces of *linearly ordered topological spaces* (LOTS), i.e., linearly ordered sets with their usual open interval topology.

2. GO-spaces with continuous separating families

If a GO-space admits a continuous separating family, then it has a continuous separating family of a special kind, as described in (2.1) below.

2.1 Proposition: *If a GO space X has a continuous separating family, then it has a continuous separating family $\{f_{x,y} : (x,y) \in S^2 - \Delta\}$ such that if $y < x$ then*
a) $f_{x,y}(t) = 0$ if $t \leq y$ and
b) $f_{x,y}(t) = 1$ if $x \leq t$.

Proof: We know from [S2] that if a space X has a continuous separating collection, then it has a continuous separating collection $\{h_{x,y} : (x,y) \in X^2 - \Delta\}$ such that $h_{x,y}(x) = -1$ and $h_{x,y}(y) = 1$. First define $g_{x,y} = \frac{1-h_{x,y}}{2}$ and note that $g_{x,y}(x) = 1$ and $g_{x,y}(y) = 0$. If $y < x$, define

$$f_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq x \\ g_{x,y}(t) & \text{if } t \geq x \end{cases}$$

In case $x < y$ define $f_{x,y} = g_{x,y}$. It is easy to see that $\{f_{x,y} : (x,y) \in X^2 - \Delta\}$ is a continuous separating family for X that satisfies conclusion (a) of this lemma. An analogous modification of $f_{x,y}$ for $t \geq y$ yields a continuous separating family that satisfies both (a) and (b). $\square$

2.2 Lemma: *Let S be a stationary set in a regular uncountable cardinal κ. Then S does not admit a continuous separating family.*

Proof: For contradiction, suppose there is a continuous separating family for S. Modify the family if necessary to make it satisfy the conclusion of (2.1).

Claim 1: For each $y \in S$ there is a $\delta(y) \in S$ such that for any $x > y$ and any $t > \delta(y)$ we have $f_{x,y}(t) = 1$.

To prove Claim 1, fix $y \in S$ and recall that the function $F(x) = f_{x,y}$ gives a continuous function from $T = S \cap]y, \kappa[$ onto a subset M of the metric space $C_u(S)$. Being a metric space, M has a σ-closed-discrete dense subset $N = \bigcup\{N(i) : i < \omega\}$. For each $g \in N$ choose $x_g \in T$ such that $g = F(x_g)$. Because F is continuous, each set $\{x_g : g \in N(i)\}$ is closed and discrete in S and therefore cannot be cofinal in S because S is stationary in κ. Then, because

κ is regular and uncountable, the set $\{x_g : g \in N\}$ cannot be cofinal in S. Choose $\delta(y) \in S$ such that $\{x_g : g \in N\} \subseteq [0, \delta(y)[$. Now fix any $x > y$ and any $t > \delta(y)$. The function $f_{x,y}$ belongs to M, so there is a sequence $\{g_n : n < \omega\}$ in N that converges to $f_{x,y}$ in M. Write $x(n)$ for the element x_{g_n} chosen above. Note that for each n, $t > \delta(y) > x(n)$ so that the special modifications of the continuous separating family in (2.1) yield $f_{x(n),y}(t) = 1$. Now, taking limits, we obtain $f_{x,y}(t) = 1$ as claimed.

For the rest of this proof, we will say that a net $\{y(\alpha) : \alpha < \lambda\}$ is *δ-admissible* provided

1) λ is a limit ordinal;

2) $y(\alpha) \in S$ for each $\alpha < \lambda$;

3) if $\alpha < \beta < \lambda$ then $y(\alpha) < \delta(y(\alpha)) < y(\beta)$.

Let C be the set of all $z < \kappa$ such that some δ-admissible net has z as its supremum.

<u>Claim 2</u>: The set C is a closed and unbounded subset of κ.

To see that C is unbounded, note that if we start at any $y(0) \in S$, we can recursively define $y(n)$ in such a way that $y(n) < \delta(y(n)) < y(n+1)$. Then $\sup\{y(n) : n < \omega\}$ belongs to C and exceeds $y(0)$.

To see that C is closed, let q be any limit point of C and choose a strictly increasing net $\{p(\alpha) : \alpha < \lambda\}$ having supremum q. For each $\alpha < \lambda$ choose a δ-admissible net $\{z(\alpha, \beta) : \beta < \beta_\alpha\}$ that converges to $p(\alpha)$. For each $\alpha < \lambda$ we have $p(\alpha) < p(\alpha + 1)$ so there must be a $\gamma = \gamma_\alpha$ with $p(\alpha) < z(\alpha + 1, \gamma) < p(\alpha + 1)$. Defining $x(\alpha) = z(\alpha + 1, \gamma)$ we obtain a δ-admissible net whose supremum is q. Thus $q \in C$, and Claim 2 is established.

Because S is stationary, there must be a point $z \in C \cap S$. Choose a δ-admissible net $\{y(\alpha) : \alpha < \mu\}$ with supremum equal to z. Let w be any point of S with $w > z$. Observe that $w > z > y(\alpha + 1) > \delta(y(\alpha)) > y(\alpha)$ for each $\alpha < \mu$.

The net $\{(w, y(\alpha)) : \alpha < \mu\}$ converges to (w, z) so that the net $\{f_{w,y(\alpha)} : \alpha < \mu\}$ converges to $f_{w,z}$. Because of the special properties of the continuous separating family established in (2.1) we know that $f_{w,z}(z) = 0$. However, the fact that $w > z > \delta(y(\alpha))$ combines with Claim 1 to show that for each α, $f_{w,y(\alpha)}(z) = 1$, and that is incompatible with $f_{w,y(\alpha)} \to f_{w,z}$. $\square$

2.3 Proposition: *Suppose that X is a GO-space (or more generally, any monotonically normal space) that has a continuous separating family. Then X is hereditarily paracompact.*

Proof: If not, then it follows form a theorem of Balogh and Rudin [BR] that X contains a topological copy of a stationary subset S of some uncountable regular cardinal κ. That subspace inherits a continuous separating family from the space X, and that is impossible in the light of (2.2). $\square$

In certain situations, Stepanova's Theorem 1.1 has an analog for GO-spaces.

2.4 Proposition: *Suppose X is a GO-space such that $X^2 - \Delta$ is either separable, or has a dense Lindelöf subspace, or has countable cellularity. Then the following are equivalent:*

(a) X has a G_δ-diagonal;

(b) X has a weaker metric topology;

(c) X has a continuous separating family.

In particular, if the GO-space X can be p-embedded in some LOTS and satisfies the hypothesis of this Proposition, then the existence of a continuous separating family is equivalent to metrizability.

Proof: The equivalence of (a) and (b) is well known for any GO-space, and (b) implies (c) for any topological space. Therefore it is enough to prove that (c) implies (b).

Suppose that $\Phi : X^2 - \Delta \to C_u(X)$ is a continuous separating family. Let $M = \Phi[X^2 - \Delta]$. Then M is a separable metric space, so there is a set $\{g_n : n \geq 0\} \subseteq M$ that is dense in M with respect to the uniform convergence topology inherited from $C_u(X)$. Fix distinct points x, y in X. Then $f_{x,y}$ separates the points x and y and some subsequence of $\{g_n : n \geq 0\}$ converges uniformly to $f_{x,y}$, so that for some n, $g_n(x) \neq g_n(y)$. Define $G : X \to \mathbb{R}^\omega$ by the rule that $G(x) = \langle g_0(x), g_1(x), g_2(x), \cdots \rangle$. Then G is a continuous, 1-1 function from X into $\mathbb{R}^\omega$, so that X has a weaker metric topology, as claimed.

Now suppose that the GO-space X can be p-embedded into some LOTS, and that $X^2 - \Delta$ is separable, or has a dense Lindelöf subspace, or has countable cellularity. Then the existence of a continuous separating family yields a G_δ-diagonal, and it is known that for GO-spaces that p-embed in some LOTS, the existence of a G_δ-diagonal yields metrizability [L]. $\square$

Remark: Two of the hypotheses in (2.4) are deceptively general. For a GO-space X, one can show that X is separable if and only if $X^2 - \Delta$ has countable cellularity. (This is essentially a result of Kurepa; see problem 2.7.10 (b) in [E].) We do not know whether the property "$X^2 - \Delta$ has a dense Lindelöf subspace" is strictly weaker than separability (for a GO-space).

Examples in Section 3 will show that Proposition (2.4) fails for more general GO-spaces and LOTS. All of the examples have uncountable cellularity and in an earlier draft of this paper, we asked whether (2.4) holds if separability of X is relaxed to countable cellularity of X. We asked

2.5 Question: Suppose that X is a non-separable GO-space that has countable cellularity (i.e., X is a Souslin space). Can X have a continuous separating family?

Recently Gary Gruenhage proved that if there is a Souslin space, then there is a Souslin space that admits a continuous separating family, thereby answering Question 2.5 in the affirmative.

By combining results of van Wouwe [vW] and Stepanova (cited above), we obtain another metrization theorem for linearly ordered spaces, namely:

2.6 Proposition: *Suppose X is a LOTS. Then X is metrizable if and only if X has a σ-closed-discrete dense subset and a continuous separating family.*

Proof: It is enough to show that every LOTS with a σ-closed-discrete dense set and a continuous separating family is metrizable. To that end, we invoke the Corollary to Theorem 2.1.6

in [vW] to see that any LOTS with a σ-closed-discrete dense subset must be a paracompact p-space in the sense of Arhangel'skii, and then the theorem of Stepanova (see (1.1), above) to conclude that X is metrizable. $\square$

Results in [BHL] allow us to extend (2.6) to perfect LOTS that have continuous separating families and certain other properties. For example:

2.7 Proposition: *Suppose X is a perfect LOTS that has a continuous separating family. Then the following are equivalent:*

a) X is metrizable;

b) X is the union of countably many metrizable subspaces;

c) X can be mapped by a continuous s-mapping onto some topological space with a G_δ-diagonal. $\square$

There is a family of questions about the difference between perfect GO-spaces and the apparently more restrictive class of GO-spaces with σ-closed-discrete dense sets. See [QT] for a survey, and a proof that many of these questions are equivalent to each other. We add another question to the list.

2.8 Question: Is there a perfect, non-metrizable LOTS that has a continuous separating family?

3. Continuous separating families and base conditions.

The examples in this section show that the existence of a continuous separating family does not yield metrizability, even in the presence of very strong base conditions, and even in the very restrictive context of linearly ordered topological spaces.

Because the Sorgenfrey and Michael lines each have a weaker metric topology (namely the usual topology of the space $\mathbb{R}$), we know that each has a continuous separating family. Thus, for GO-spaces in general, Stepanova's continuous separating families do not yield metrizability. However, experience shows that it can happen that a topological property does not yield metrizability for GO-spaces, but does imply metrizability for linearly ordered spaces. One such property is the existence of a G_δ-diagonal. Examples in this section show that Stepanova's continuous separating families do not yield metrizability, even in linearly ordered spaces.

3.1 Example *There is a LOTS with a continuous separating family that is not first countable.*

Proof: Let X be the space obtained by isolating every countable ordinal in $[0, \omega_1]$ and letting ω_1 have its usual neighborhoods. Then X is homeomorphic to the lexicographic product $([0, \omega_1[\times \mathbb{Z}) \cup \{(\omega_1, 0)\}$, so X is a linearly ordered topological space. For any $(\alpha, \beta) \in X^2 - \Delta$ let $\Phi(\alpha, \beta)$ be the characteristic function of the interval $[0, \min(\alpha, \beta)]$. Then Φ gives the required continuous separating family. $\square$

It is harder to find *first-countable* linearly ordered spaces that have continuous separating families and are not metrizable. The LOTS extension of the Michael line is one such space.

3.2 Example: *There is a first countable LOTS that has a σ-disjoint base and has a continuous separating family, and is not metrizable.*

Proof: Let $X = M^*$ be the LOTS extension of the Michael line, i.e., let X be the set

$$(\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \mathbb{Z})$$

equipped with the open interval topology of the lexicographic order. Then X is a LOTS and has a σ-disjoint base [L]. To show that X has a continuous separating family, we define $\Phi((x,i),(y,j)) = f_{(x,i),(y,j)}$ where

a) $f_{(x,i),(y,j)}((z,k)) = |x - z|$ in case $x \neq y$

b) $f_{(x,i),(x,j)}$ is the characteristic function of the set $\{(x,j)\}$ in case $x = y$.

Then $f_{(x,i),(y,j)}((x,i)) \neq f_{(x,i),(y,j)}((y,j))$ and each $f_{(x,i),(y,j)}$ is continuous from X to $\mathbb{R}$

To show that Φ is continuous, suppose $((x_n,i_n),(y_n,j_n))$ is a sequence in $X^2 - \Delta$ that converges to $((x_0,i_0),(y_0,j_0)) \in X^2 - \Delta$. We must show that the sequence $f_{(x_n,i_n),(y_n,j_n)}$ converges uniformly to $f_{(x_0,i_0),(y_0,j_0)}$. We consider four cases, depending upon the nature of the points x_0 and y_0.

Case 1: If $x_0, y_0 \in \mathbb{P}$ then the sequence $((x_n,i_n),(y_n,j_n))$ is eventually constant because both (x_0,i_0) and (y_0,j_0) are isolated points, so there is nothing to prove.

Case 2: Suppose $x_0 \in \mathbb{P}$ and $y_0 \notin \mathbb{P}$. Then (x_0,i_0) is isolated and we may assume that $(x_n,i_n) = (x_0,i_0)$ for all n. Further, $y_0 \in \mathbb{R} - \mathbb{P}$ forces $j_0 = 0$. Because $x_0 \neq y_0$, we may assume that for each n, $(y_n,j_n) \neq (x_0,i_0) = (x_n,i_n)$. Therefore, for each (z,k), $f_{(x_n,i_n),(y_n,j_n)}((z,k)) = |x_n - z|$ and $f_{(x_0,i_0),(y_0,j_0)}((z,k)) = |x_n - z|$. Hence $x_n \to x_0$ shows that we have the required uniform convergence.

Case 3: Suppose $x_0 \notin \mathbb{P}$ (whence $i_0 = 0$) and $y_0 \in \mathbb{P}$. This case is analogous to Case 2.

Case 4: Suppose neither x_0 nor y_0 belongs to $\mathbb{P}$. Then $i_0 = 0 = j_0$ so that $((x_0,i_0),(y_0,j_0)) \in X^2 - \Delta$ forces $x_0 \neq y_0$. Because $x_n \to x_0$ and $y_n \to y_0$ we may assume that every x_n lies on the same side of $\frac{x_0+y_0}{2}$ as does x_0, and that each y_n lies on the same side of $\frac{x_0+y_0}{2}$ as does y_0. Thus, $x_n \neq y_n$ for each n. But then $f_{(x_n,i_n),(y_n,j_n)}((z,k)) = |x_n - z|$ and $f_{(x_0,i_0),(y_0,j_0)}((z,k)) = |x_0 - z|$ so once again we have the required uniform convergence. $\square$

Remark: In a recent paper, [HH], Halbeisen and Hungerbuhler proved that a space X admits a continuous separating family that depends on only one parameter (in the sense that $f_{x,y} = f_{x,z}$ whenever $y,z \in X - \{x\}$) if and only of X has a weaker metric topology. They gave an example of a paracompact space S that has a continuous separating family, but does not have a one-parameter continuous separating family. Their example had $\chi(S) = \omega_1$. Because the LOTS M^* is non-metrizable, it cannot have a G_δ-diagonal and hence does not have a weaker metric topology. It follows from (3.2) and the characterization in [HH] that M^* is another example of a space that has a continuous separating family but does not have a one-parameter continuous separating family, and M^* has the advantage that it is also first countable.

3.3 Example: *There is a Lindelöf LOTS X with a σ-disjoint base that has a continuous separating family and is not metrizable.*

Proof: Let $B \subset [0,1]$ be a Bernstein set, i.e., a set such that if K is an uncountable compact subset of $[0,1]$, then both $K \cap B$ and $K - B$ are non-empty. Bernstein sets exist in ZFC [Ox]. Write $C = [0,1] - B$ and define $X = (B \times \mathbb{Z}) \cup (C \times \{0\})$. Let X have the open interval topology of the lexicographic order. The resulting LOTS is Lindelöf and is not metrizable, and has a σ-disjoint base. A continuous separating family $\{f_{x,y} : (x,y) \in X^2 - \Delta\}$ for X can be defined as in in (3.2)

It is interesting to note that the space M^* of (3.2) consistently contains Lindelöf subspaces that inherit a continuous separating family from M^* and are non-metrizable. In [M1], Michael showed that CH yields a subset L of the usual space of real numbers that contains $\mathbb{Q}$ and is concentrated on $\mathbb{Q}$. Burke and Davis [BD] used the existence of an ω_1-scale to get the same conclusion. Topologize L as a subspace of the Michael line M and let $X = L^*$ be the LOTS extension of L, i.e., the lexicographically ordered set

$$(\mathbb{Q} \times \{0\}) \cup ((L - \mathbb{Q}) \times \mathbb{Z}).$$

This space is known to be Lindelöf and non-metrizable and it inherits a σ-disjoint base and a continuous separating family from the superspace M^*. $\square$

While the spaces described in (3.2) and (3.3) are not metrizable, they do have a σ-disjoint base, and that is a very strong base condition among ordered spaces. Our next example shows that the existence of a continuous separating family does not yield a point-countable base and does not force a point-countable base to be σ-disjoint.

3.4 Example: *There is a LOTS with a continuous separating family that does not have a point-countable base, and there is a LOTS with a continuous separating family that has a point-countable base but not a σ-disjoint base.*

Proof: For each $\alpha \leq \omega_1$, let $B_\alpha = \{x : [0,\alpha] \to \mathbb{R} : \forall \beta < \alpha,\ x(\beta) \in \mathbb{P} \text{ and } x(\alpha) \in \mathbb{Q}\}$. Let $B = \bigcup\{B_\alpha : \alpha \leq \omega_1\}$. Note that if x and y are distinct elements of B, then there is a first ordinal $\alpha = fd(x,y)$ such that $x(\alpha) \neq y(\alpha)$. We define $x \prec y$ if $x(\alpha) < y(\alpha)$ and $y \prec x$ otherwise. Endow B with the open interval topology of the order $\prec$. One can verify that basic neighborhoods of $x \in B_\alpha$ have the form $B(x,n) = \{y \in B : y(\beta) = x(\beta) \text{ if } \beta < \alpha \text{ and } |y(\alpha) - x(\alpha)| < \frac{1}{n}\}$. In [BL] we showed that the LOTS B does not have a point-countable base, and in [B] it was shown that the subspace $C = \bigcup\{B_\alpha : \alpha < \omega_1\}$ is a LOTS with a point-countable base, but not a σ-disjoint base. We will complete this proof by showing that the LOTS B has a continuous separating family $\{f_{x,y} : (x,y) \in B^2 - \Delta\}$. Then the subcollection $\{f_{x,y}|C : (x,y) \in C^2 - \Delta\}$ will be a continuous separating family for the subspace C.

It is enough to define $f_{x,y}$ in case $x \prec y$ in B. Compute $\alpha = fd(x,y)$ and note that $x(\alpha) < y(\alpha)$. For any $t \in B$, define

$$f_{x,y}(t) = \begin{cases} 0 & \text{if } t \preceq x \\ 1 & \text{if } y \preceq t \\ \frac{t(\alpha)-x(\alpha)}{y(\alpha)-x(\alpha)} & \text{if } x \preceq t \preceq y. \end{cases}$$

A case-by-case analysis shows that if $\langle(x_n, y_n)\rangle$ converges to (x, y) in $B^2 - \Delta$, then $\langle f_{x_n,y_n}\rangle$ converges uniformly to $f_{x,y}$ in $C_u(B)$. $\square$

A LOTS that is, in many senses, at the opposite end of the spectrum from M^* is the LOTS extension S^* of the Sorgenfrey line S. Recall that S^* is the lexicographically ordered set $\mathbb{R} \times \{n \in \mathbb{Z} : n \leq 0\}$. Whether or not S^* has a continuous separating family may be axiom-sensitive. The hypothesis of our next result – that there is an uncountable subspace of S whose square is Lindelöf – is a consequence of the Continuum Hypothesis [M1] and is consistent with (MA + not CH), but is not consistent with OCA or PFA [BuMo].

3.5 Example: *If there is an uncountable subspace of the Sorgenfrey line S whose square is Lindelöf, then S^* does not have a continuous separating family.*

Proof: Let T be an uncountable subspace of the Sorgenfrey line S such that $T \times T$ is Lindelöf. A result of Michael [HM] shows that S^2 is perfect, so we know that T^2 is hereditarily Lindelöf.

Let $\preceq$ be the lexicographic order on S^*. For contradiction, suppose that there is a continuous separating family for S^*, say $\{f_{(x,i),(y,j)} : ((x, i), (y, j)) \in (S^*)^2 - \Delta\}$. By Lemma (2.1), we may assume that if $(y, j) \prec (x, i)$, then

$$f_{(x,i),(y,j)}((z, k)) = \begin{cases} 0 \text{ whenever } (z, k) \preceq (y, j) \\ 1 \text{ whenever } (y, j) \preceq (z, k). \end{cases}$$

Consider the subspace $X = \{(x, i) \in S^* : x \in T, i \in \{0, 1\}\}$ of S^*. Restricting the continuous separating family to the subspace X gives a continuous separating family for X. The rest of the proof is devoted to using the special properties of the Cartesian product space T^2 to show that the space X cannot have a continuous separating family.

Fix k and let $B_k = \{(x, y) \in T^2 : y < x \text{ and } |f_{(x,0),(y,-1)}((y, 0))| \geq \frac{1}{k}\}$. We claim that the set $Y_k = \{y \in T : (T \times \{y\}) \cap B_k \neq \emptyset\}$ is countable. If not, choose distinct $y_\alpha \in Y_k$ for $0 \leq \alpha < \omega_1$ and then choose x_α such that $(x_\alpha, y_\alpha) \in B_k$. As noted above, the space T^2 is hereditarily Lindelöf, so that the uncountable set $A = \{(x_\alpha, y_\alpha) : \alpha < \omega_1\}$ must contain a limit point of itself, say (x', y'). Choose ordinals $\alpha_n < \omega_1$ such that $(x_{\alpha_n}, y_{\alpha_n})$ converges to (x', y'). Because the points y_α are distinct, the sequence $\{y_{\alpha_n} : n \geq 1\}$ has a strictly monotone decreasing subsequence. We may assume that $y_{\alpha_n} > y_{\alpha_{n+1}}$ for each n. Then we have

(a) $\lim_{n\to\infty}(x_{\alpha_n}, 0) = (x', 0)$

(b) $\lim_{n\to\infty}(y_{\alpha_n}, 0) = (y', 0)$

(c) $\lim_{n\to\infty}(y_{\alpha_n}, -1) = (y', 0)$

(d) because $(x', y') \in A \subseteq B_k$, we know that $y' < x'$.

Then the sequence $\langle f_{(x_{\alpha_n},0),(y_{\alpha_n},-1)}\rangle$ converges to $f_{(x',0),(y',0)}$ uniformly and therefore

$$\lim_{n\to\infty} f_{(x_{\alpha_n},0),(y_{\alpha_n},-1)}((y_{\alpha_n}, 0)) = f_{(x',0),(y',0)}(y', 0) = 0.$$

However $|f_{(x_{\alpha_n},0),(y_{\alpha_n},-1)}((y_{\alpha_n},0))| \geq \frac{1}{k}$ for each n, and that is impossible. Therefore, each set Y_k is countable, and hence so is the set $Y = \bigcup\{Y_k : k \geq 1\}$.

Because T is uncountable and hereditarily Lindelöf, the set T contains an ω_1-limit point $\hat{y}$ of itself, i.e., a point $\hat{y}$ such that $T \cap]\hat{y}, \hat{y} + \frac{1}{n}[$ is uncountable for each n. Indeed, T must contain uncountably many such points, so we may choose one that is not in the countable set Y. We will denote the chosen point by $\hat{y}$. For each n, choose $x_n \in (]\hat{y}, \hat{y} + \frac{1}{n}[\cap T) - Y$. Then $x_n > \hat{y}$ and because $\hat{y} \notin Y$ no point $x \in T \cap]\hat{y}, \rightarrow[$ can have $f_{(x,0),(\hat{y},-1)}(\hat{y}, 0) \neq 0$. In particular, for all n, we have $f_{(x_n,0),(\hat{y},-1)}((\hat{y}, 0)) = 0$.

But $\langle x_n \rangle \to \hat{y}$ so that in L, $(x_n, 0) \to (\hat{y}, 0)$. Consequently $f_{(x_n,0),(\hat{y},-1)}$ converges uniformly to $f_{(\hat{y},0),(\hat{y},-1)}$ and hence $f_{(x_n,0),(\hat{y},-1)}((\hat{y}, 0))$ converges to $f_{(\hat{y},0),(\hat{y},-1)}((\hat{y}, 0))$. But that is impossible because

$$f_{(x_n,0),(\hat{y},-1)}((\hat{y}, 0)) = 0$$

for every n while

$$f_{(\hat{y},0),(\hat{y},-1)}((\hat{y}, 0)) = 1.$$

That contradiction completes the proof of (3.4) $\square$

3.6 Question: In ZFC, does S^* have a continuous separating family?

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