Cleavability in Ordered Spaces

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Abstract. In this paper we study the role of cleavability and divisibility in the topology of generalized ordered (GO-)spaces. We characterize cleavability of a GO-space over the class of metrizable spaces, and over the spaces of irrational and rational numbers. We present a series of examples related to characterizations of cleavability over separable metric spaces and over the space of real numbers.

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1. Introduction

Recall that a *linearly ordered topological space* (LOTS) is a linearly ordered set equiped with the usual open-interval topology of the given order. By a *generalized ordered space* (GO-space) we mean a triple $(X, \mathcal{T}, <)$ where $<$ is a linear ordering of the set X and where $\mathcal{T}$ is a Hausdorff topology on X that has an base of open sets that are order-convex. It is well-known that a subspace of a LOTS may fail to be a LOTS and that the class of GO-spaces is exactly the class of all subspaces of linearly ordered spaces.

In this paper we study the role of cleavability and divisibility in the class of linearly ordered and generalized ordered spaces. Let $\mathcal{M}$ be the class of metrizable spaces and let $\mathcal{S}$ be the class of separable metrizable spaces. To say that a topological space X is *cleavable over* $\mathcal{M}$ means that for every subset $A \subseteq X$, there is a space $M_A \in \mathcal{M}$ and a continuous function $f_A : X \to M_A$ such that if $x \in A$ and $y \in X - A$, then $f_A(x) \neq f_A(y)$. The function f_A is called a *cleaving function for* A. The term *cleavability over* $\mathcal{S}$ is analogously defined. Cleavability over $\mathcal{M}$ and over $\mathcal{S}$ generalize the classical properties "there is a continuous 1-1 mapping from X into a member of $\mathcal{M}$ (respectively, a member of $\mathcal{S}$)" which are now called *absolute cleavability over* $\mathcal{M}$ (respectively *over* $\mathcal{S}$). These properties have been studied extensively [Ar, Ar2], as have the analogously defined notions of cleavability over the spaces $\mathbb{R}$, $\mathbb{P}$, and $\mathbb{Q}$ of real, irrational, and rational numbers, respectively [Ar3].

A property related to cleavability over $\mathcal{S}$ is divisibility, where we say that a topological space X is *divisible* if for each $A \subseteq X$ there is a *countable* collection $\mathcal{D}$ of open subsets of X such that if $x \in A$ and $y \in X - A$, then some $D \in \mathcal{D}$ has $x \in D \subseteq X - \{y\}$. The collection $\mathcal{D}$ is called a *divisor for* A and one can show (4.1) that a topological space is cleavable over $\mathcal{S}$ if and only if each subset of X has a divisor that consists of cozero sets of continuous, real-valued functions.

Restricting attention to the class of generalized ordered spaces allows us to characterize several cleavability properties in more classical terms, and raises some interesting questions. In Section 3 we show that, among GO-spaces, cleavability over $\mathcal{M}$ is the same as absolute cleavability over $\mathcal{M}$ and is equivalent to several other classical properties. It will follow that a LOTS X is cleavable over $\mathcal{M}$ if and only if $X \in \mathcal{M}$, and that no Souslin space (a LOTS with cellularity $= \omega <$ density) can be cleavable over $\mathcal{M}$.

In (4.1) we point out several characterizations of cleavability over $\mathcal{S}$ for general topological spaces (e.g., that a topological space is cleavable over $\mathcal{S}$ if and only if it is divisible by cozero sets), but we do not have any characterizations of cleavability over $\mathcal{S}$ that utilize the special structure of GO-spaces.

1.1 Open Problem: *Characterize GO-spaces that are divisible; characterize GO-spaces that are cleavable over $\mathcal{S}$.*

While we have some necessary conditions (see (4.3) and (4.4)), we know that those conditions are not sufficient. However we do have characterizations (see 4.7) of GO-spaces that are absolutely cleavable over $\mathcal{S}$, e.g., that a space X is absolutely cleavable over $\mathcal{S}$ if and only if X is cleavable over $\mathcal{S}$ and has $c(X) \leq \mathbf{c}$. (Throughout this paper, we use the symbol $\mathbf{c}$ to denote the cardinal 2^ω and $c(X)$ to denote the cellularity of a space X.) Furthermore, for GO-spaces, absolute cleavability over $\mathcal{S}$ is characterized by the existence of a countable, point-separating cover by cozero sets.

We do not have a characterization of cleavability over $\mathbb{R}$ for GO-spaces. As discussed in (5.3), Examples (4.6) and (5.2) can be used to give necessary conditions for a GO-space to be cleavable over $\mathbb{R}$, but those conditions are not sufficient, and we have:

1.2 Open Problem: *Characterize GO-spaces that are cleavable over $\mathbb{R}$.*

In (5.5) we characterize GO-spaces that are cleavable over $\mathbb{P}$, showing that they are exactly the GO-spaces that are cleavable over $\mathcal{S}$ and have $ind(X) = 0$. Equivalently, they are the GO-spaces in which every $A \subset X$ has a countable clopen divisor, i.e., a countable collection $\mathcal{C}$ of clopen subsets of X such that if $x \in A \subseteq X - \{y\}$, then some $C \in \mathcal{C}$ has $x \in C \subseteq X - \{y\}$.

In (2.3) we give an easy characterization of absolute cleavability over $\mathbb{P}$ that applies to any topological space, and that result allows us to exhibit some strange linearly ordered spaces that are absolutely cleavable over $\mathbb{P}$, e.g., A.H. Stone's metric space (5.7) and the lexicographic product space $[0, \mathbf{c}[\times \mathbb{P}$, and more generally any metric space X with cardinality $\leq c$ and $Ind(X) = 0$. As for cleavability over $\mathbb{Q}$, we show in (5.9) that a GO-space X is cleavable over $\mathbb{Q}$ if and only if each subset of X is an F_σ-subset of X.

Finally, we give examples that distinguish between various types of cleavability among GO-spaces. We show that a GO-space can be:

a) (absolutely) cleavable over $\mathcal{M}$ but not metrizable (e.g., both the Sorgenfrey line and the Michael line);

b) metrizable (and hence cleavable over $\mathcal{M}$) but not cleavable over $\mathcal{S}$ (e.g., the lexicographic product $[0, c^+[\times \mathbb{R})$;

c) (absolutely) cleavable over $\mathcal{S}$ but not cleavable over $\mathbb{R}$ (e.g., the lexicographic product $[0, \omega_1[\times \mathbb{R})$ (see (5.2));

d) cleavable over $\mathbb{R}$ but not over $\mathbb{P}$ (e.g., the space $\mathbb{R}$ itself);

e) cleavable over $\mathbb{P}$ but not over $\mathbb{Q}$ (e.g., the space $\mathbb{P}$ – see (5.9)).

2. Special subsets of $(X, <)$ and Kowalsky-type lemmas

We begin this section with two lemmas that construct four special subsets A, B, C, and D of any linearly ordered set $(X, <)$. These lemmas will be the key to later characterization theorems. Two points x, y of a linearly ordered set $(X, <)$ are *adjacent* if one is the immediate successor of the other. Let N be the set of all *neighbor points* of X, i.e., $N = \{x \in X : x \text{ is adjacent to some point } y \in X\}$.

2.1 Lemma: *Let N be the set of all neighbor points in a linearly ordered set $(X, <)$. Then there are sets $A, B \subseteq X$ such that:*

a) $A \cup B = N$ and $A \cap B = \emptyset$;

b) if $x \in A$ and if $y \in X$ is adjacent to x, then $y \in B$;

c) *if $x \in B$ and if $y \in X$ is adjacent to x, then $y \in A$.*

Proof. For any subset $S \subseteq X$, let $conv(S)$ denote the convex hull of S in X. For $x, y \in X$, define $x \sim y$ to mean that $conv(\{x, y\})$ is finite. Then $\sim$ is an equivalence relation on X. Denote the equivalence class of $\sim$ to which x belongs by $[x]$ and let $\mathcal{N} = \{[x] : x \in X \text{ and } |[x]| \geq 2\}$. Observe that $N = \bigcup \mathcal{N}$. Depending upon its endpoint structure, each $[x] \in \mathcal{N}$ looks like a strictly monotone sequence, or else a copy of the set $\mathbb{Z}$ of all integers.

Let $\mathcal{L} = \{[x] \in \mathcal{N} : [x] \text{ contains a left endpoint}\}$. For each $[x] \in \mathcal{L}$ there is a strictly increasing function $f_{[x]}$ whose domain $dom(f_{[x]})$ is an initial segment of, or perhaps all of, $[0, \omega[$ and whose range is all of $[x]$. (Note that if $2 \leq |[x]| < \omega$, then $[x] \in \mathcal{L}$.)

Let $\mathcal{R} = \{[x] \in \mathcal{N} - \mathcal{L} : [x] \text{ contains a right endpoint}\}$. Then for each $[x] \in \mathcal{R}$ there is a strictly decreasing function $f_{[x]}$ whose domain is $[0, \omega[$ and whose range is all of $[x]$.

If $[x] \in \mathcal{N} - (\mathcal{L} \cup \mathcal{R})$ there is a strictly increasing function $f_{[x]}$ whose domain is the usual set of all integers, and whose range is all of $[x]$.

Now let $A = \{f_{[x]}(n) : [x] \in \mathcal{N}, n \in dom(f_{[x]}) \text{ and } n \text{ is even}\}$, and let $B = \{f_{[x]}(n) : [x] \in \mathcal{N}, n \in dom(f_{[x]}) \text{ and } n \text{ is odd}\}$. Then $A \cup B = N$ and $A \cap B = \emptyset$, and it is clear that neither A nor B can contain adjacent points of $(X, <)$. □

2.2 Lemma: *Suppose $(X, <)$ is an infinite linearly ordered set. Then there exist sets $C, D \subseteq X$ satisfying:*

a) *$C \cap D = \emptyset$ and $C \cup D = X$;*

b) *if $u < v$ are points of X and if $]u, v[$ is infinite, then $]u, v[\cap C \neq \emptyset$ and $]u, v[\cap D \neq \emptyset$.*

Proof. We will say that a pair (A, B) of subsets of X is interlaced provided $]a_1, a_2[\cap B \neq \emptyset$ whenever $a_1 < a_2$ are points of A, and $]b_1, b_2[\cap A \neq \emptyset$ whenever $b_1 < b_2$ are points of B. Because X is an infinite linearly ordered set, X contains a strictly monotonic sequence $\{x_n : n \geq 1\}$ of points. Dividing those points into two sets based upon the parity of n yields an interlaced pair of disjoint subsets of X. Hence we see that the collection $\mathcal{P} = \{(A, B) : A \cap B = \emptyset \text{ and } (A, B) \text{ is an interlaced pair of subsets of } X\}$ is nonempty. Partially order $\mathcal{P}$ by $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. We apply Zorn's Lemma to $\mathcal{P}$ to choose a maximal element $(C, D) \in \mathcal{P}$.

To verify b), suppose $u < v$ are points of X such that $]u, v[$ is infinite. For contradiction, suppose that $C \cap]u, v[= \emptyset$. Then $D \cap]u, v[$ has at most one point, because if $d_1 < d_2$ were two points of that set, then by interlacing, some point of C would lie between d_1 and d_2, forcing $C \cap]u,v[\neq \emptyset$. If $D \cap]u,v[$ is nonempty, remove its unique point w and replace $]u,v[$ by whichever of $]u,w[$ and $]w,v[$ is infinite. Thus we may assume that $D \cap]u,v[= \emptyset$.

Because $]u,v[$ is infinite, it contains an infinite strictly monotonic sequence $\{x_n : n \geq 1\}$. Consider the case where $x_n < x_{n+1}$ for each n, the other case being analogous. There are several cases to consider based upon the endpoint structure of the sets $C \cap]\leftarrow, u[$ and $D \cap]\leftarrow, u[$.
Case 1: Suppose that $C \cap]\leftarrow, u[$ has no largest point and yet $D \cap]\leftarrow, u[$ does have a largest point. Then define $C_1 = C \cup \{x_n : n \geq 1$ and n is odd $\}$ and $D_1 = D \cup \{x_n : n \geq 1$ and n is even $\}$. Then $(C_1, D_1) \in \mathcal{P}$ contradicting maximality of (C, D).
Case 2: Suppose that $D \cap]\leftarrow, u[$ has no largest point and yet $C \cap]\leftarrow, u[$ does have a largest point. This case parallels Case 1.
Case 3: Suppose that both $C \cap]\leftarrow, u[$ and $D \cap]\leftarrow, u[$ have largest points, say c_0 and d_0 respectively. Consider the subcase where $c_0 < d_0$, the other subcase being analogous. Define C_1 and D_1 as in Case 1, once again contradicting maximality of (C, D).
Case 4: Suppose that neither $C \cap]\leftarrow, u[$ nor $D \cap]\leftarrow, u[$ has a largest point. Defining C_1, D_1 as in Case 1, we once again violate maximality of (C, D).

Those four cases show that $C \cap]u,v[= \emptyset$ is impossible. Similarly, $D \cap]u,v[= \emptyset$ is also impossible. As a final step in the proof, replace the set D by the larger set $X - C$. The resulting pair (C, D) satisfies both (a) and (b) above. □

We conclude this section with a collection of characterizations of various kinds of absolute cleavability (defined in the Introduction). Recall Kowalsky's proof that any metric space embeds in a countable product of suitably spiny hedgehog metric spaces ([E,4.4.9] and [Kw]). A slight variation on that proof yields the following results, which must be well-known.

2.3 Proposition: *Let X be any topological space. Then:*

a) *there is a continuous 1-1 mapping from X into a metric space if and only if there is a σ-discrete collection $\mathcal{C}$ of cozero subsets of X with the property that, given distinct points x, y of X, some $C \in \mathcal{C}$ contains exactly one of x and y;*

b) *there is a continuous 1-1 mapping from X into a separable metric space if and only if there is a countable collection $\mathcal{D}$ of cozero subsets of X that separates points of X as in (a);*

c) *there is a continuous 1-1 mapping from X into $\mathbb{P}$ if and only if $|X| \leq \mathbf{c}$ and there is a σ-discrete collection $\mathcal{E}$ of clopen subsets of X that separates points of X as described in (a).*

d) *there is a continuous 1-1 mapping from X into $\mathbb{P}$ if and only if there is a countable collection $\mathcal{F}$ of clopen subsets of X that separates points of X as described in (a);*

Proof. We will sketch proofs of (c) and (d), the proofs of (a) and (b) being analogous and easier.

Consider (c). If there is a continuous 1-1 mapping from X into $\mathbb{P}$, then we obtain the required point-separating clopen collection in X by using inverse images of a countable clopen base for $\mathbb{P}$. To prove the converse assertion, write the σ-discrete collection $\mathcal{E}$ as $\bigcup\{\mathcal{E}_n : n \in \omega\}$ where each collection $\mathcal{E}_n$ is a discrete collection of clopen sets in X. Because $|X| \leq \mathbf{c}$, we have $\kappa_n = |\mathcal{E}_n| \leq c$ for each n. Index $\mathcal{E}_n$ without repetitions as $\{E(n, \alpha) : \alpha < \kappa_n\}$.

Fix $n \in \omega$ and choose a family $\{L_\alpha : \alpha < \kappa_n\}$ of distinct lines in the plane, each passing through the point (π, π). Choose a point $p_\alpha \in (L_\alpha \cap \mathbb{P}^2)$ lying at least one unit from (π, π). Such a choice is possible because L_α contains at most countably many points with either coordinate rational. Now define $f_n : X \to \mathbb{P}^2$ by the rule that $f_n(x) = (\pi, \pi)$ if $x \in X - \bigcup \mathcal{E}_n$ and $f_n(x) = p_\alpha$ if $x \in E(n, \alpha)$ for some $\alpha < \kappa_n$. Because $\mathcal{E}_n$ is a discrete collection in X, the function f_n is continuous. Define $f : X \to (\mathbb{P}^2)^\omega$ by $f(x) = \langle f_0(x), f_1(x), f_2(x), \cdots \rangle$. Then f is continuous and the point-separating property of $\mathcal{E}$ forces f to be 1-1. But we know that $(\mathbb{P}^2)^\omega$ is homeomorphic to $\mathbb{P}$, so the proof of c) is complete.

To prove (d), suppose that there is a 1-1 continuous function from X into $\mathbb{P}$ and use the inverse images of members of a countable clopen base for $\mathbb{P}$ to get the required point-separating collection in X. Conversely, given the countable, clopen, point-separating collection $\mathcal{F} = \{F_n : n \in \omega\}$, for each n let g_n be the characteristic function of F_n, viewed as a function from X into the usual set $\mathbb{Z}$ of integers, and define $g : X \to \mathbb{Z}^\omega$ to be the function with $g(x) = \langle g_0(x), g_1(x), g_2(x), \cdots \rangle$. Then g is continuous and 1-1, and because $\mathbb{Z}^\omega$ is homeomorphic to $\mathbb{P}$, the proof is complete. □

3. Cleavability over $\mathcal{M}$

Our next result shows that for GO-spaces, cleavability over $\mathcal{M}$ is equivalent to several familiar properties.

3.1 Theorem: *Let $(X, \mathcal{T}, <)$ be a GO-space. The following are equivalent:*

a) *X has a G_δ-diagonal;*

b) *there is a metrizable topology $\mathcal{T}_m \subseteq \mathcal{T}$ on X such that $(X, \mathcal{T}_m, <)$ is a GO-space;*

c) *there is a metrizable topology $\mathcal{T}_m$ on X with $\mathcal{T}_m \subseteq \mathcal{T}$;*

d) *there is a continuous 1-1 mapping from $(X, \mathcal{T})$ into a metric space, i.e., X is absolutely cleavable over $\mathcal{M}$;*

e) *$(X, \mathcal{T})$ is cleavable over $\mathcal{M}$;*

f) *$(X, \mathcal{T})$ is cleavable over a class of spaces having a G_δ-diagonal;*

g) *there is a σ-discrete collection $\mathcal{C}$ of cozero subsets of X such that if x, y are distinct points of X then for some $C \in \mathcal{C}$, $|C \cap \{x, y\}| = 1$.*

Proof. That a) $\Rightarrow$ b) in a GO-space is a result of Przymusinski (see [Al]). Clearly b) $\Rightarrow$ c) $\Rightarrow$ d) $\Rightarrow$ e) $\Rightarrow$ f). Thus it remains only to prove that f) $\Rightarrow$ a) and that c) is equivalent to g). That f) implies a) is in Proposition 3.2 below, and the equivalence of c) and g) is in Proposition (2.3-a), above. $\square$

3.2 Proposition: *Let $(X, \mathcal{T}, <)$ be a GO-space. If $(X, \mathcal{T})$ is cleavable over a class of spaces each having a G_δ-diagonal, then $(X, \mathcal{T})$ has a G_δ-diagonal.*

Proof. For any point x and for any collection $\mathcal{L}$ of sets, we write $St(x, \mathcal{L}) = \bigcup\{L \in \mathcal{L} : x \in L\}$. Let A, B, C and D be the four subsets of X found in Lemmas (2.1) and (2.2). Let $\mathcal{E} = \{A, B, C, D\}$. For each $E \in \mathcal{E}$, choose a space M_E with a G_δ-diagonal and a continuous function $f_E : X \rightarrow M_E$ with the property that if $x \in E$ and $y \notin E$, then $f_E(x) \neq f_E(y)$. Choose a G_δ-diagonal sequence $\langle \mathcal{H}_E(n) \rangle$ of open covers of M_E. Define $\mathcal{G}_E(n) = \{G \subseteq X : G$ is a convex component of $f_E^{-1}[H]$ for some $H \in \mathcal{H}_E(n)\}$. Each $\mathcal{G}_E(n)$ is an open cover of X. We will show that the collection $\{\mathcal{G}_E(n) : n \geq 1, E \in \mathcal{E}\}$ is a G_δ-diagonal sequence for the space X.

To that end, suppose x and y are distinct points of X. We may assume $x < y$. There are several cases to consider.
<u>Case 1</u>: Consider the case where $|E \cap \{x, y\}| = 1$ for some $E \in \mathcal{E}$. We may assume that $x \in E$ and $y \notin E$. Then $f_E(x) \neq f_E(y)$ in the space M_E so that there is an n such that no single member of $\mathcal{H}_E(n)$ contains

both $f_E(x)$ and $f_E(y)$. Let $G \in \mathcal{G}_E(n)$ with $x \in G$. If $y \in G$, then $f_E(x)$ and $f_E(y)$ both belong to a single member of $\mathcal{H}_E(n)$ and that is impossible by our choice of n. Hence $y \notin St(x, \mathcal{G}_E(n))$ as required.

Having completed the proof if Case 1 holds, for the rest of this proof we will assume that Case 1 does not apply. That is, we will assume:

(*) for each $E \in \mathcal{E},\ x \in E$ if and only if $y \in E$.

It remains to consider two cases, depending upon whether the interval $[x, y]$ is finite or infinite.

<u>Case 2</u>: Suppose (*) holds and $[x, y]$ is finite. But then $[x, y] \subseteq N$ where N is the set of neighbor points defined in Section 2. By (2.1), $N = A \cup B$ so that we may assume $x \in A$. If $]x, y[= \emptyset$ then x and y are adjacent so that $y \in B$ (see (2.1)). But then $y \notin A$ and that is impossible in the light of (*). Therefore $]x, y[$ is non-empty. Next observe that $]x, y[\subseteq A$ is impossible in the light of (2.1) because the non-empty finite set $]x, y[$ must contain an immediate successor of x, and that point must lie in B. So choose any $z \in]x, y[$ with $z \notin A$. Then $f_A(x) \neq f_A(z)$ so there must be an n such that no single member of $\mathcal{H}_A(n)$ contains both $f_A(x)$ and $f_A(z)$. Consider any $G \in \mathcal{G}_A(n)$ that contains x. If $y \in G$, then convexity of G would yield $z \in [x, y] \subseteq G$ so that a single member of $\mathcal{H}_A(n)$ contains both $f_A(x)$ and $f_A(z)$ and that is impossible by our choice of n. Thus, in Case 2 we have $y \notin St(x, \mathcal{G}_A(n))$ as required.

<u>Case 3</u>: Suppose (*) holds and the interval $[x, y]$ is infinite. Then so is the interval $]x, y[$ so that, by (2.2), we may choose $p \in C \cap]x, y[$ and $q \in D \cap]x, y[$. Without loss of generality, suppose $p < q$. Because $p \in C$ and $q \notin C$, we know that $f_C(p) \neq f_C(q)$ so that for some n, no member of $\mathcal{H}_C(n)$ contains both $f_C(p)$ and $f_C(q)$. Now choose any member $G \in \mathcal{G}_C(n)$ that contains x. If $y \in G$ then convexity of G would yield $p, q \in [x, y] \subseteq G$ so that $f_C(p)$ and $f_C(q)$ would both belong to a single member of $\mathcal{H}_C(n)$ contrary to our choice of n. Hence $y \notin St(x, \mathcal{G}_C(n))$ as required. $\square$

3.3 Corollary: *If X is a LOTS that is cleavable over $\mathcal{M}$, then X is metrizable.*

Proof. Such a space has a G_δ-diagonal (3.2) and any LOTS with a G_δ-diagonal is metrizable [L1]. $\square$

3.4 Remark: It follows from (3.3) that no linearly ordered Souslin space can be cleavable over $\mathcal{M}$. A slightly more complicated argument shows that no GO-space X can have countable cellularity, uncountable density, and still be cleavable over $\mathcal{M}$. Such a GO-space would have a G_δ-diagonal (3.2) and would therefore have a dense metrizable subspace Y [BLP, Proposition 3.4]. Then Y would also have countable cellularity, so Y would be separable. Hence X is also separable. Finally, we note that an assertion in [K2] that a GO-space is metrizable provided it is cleavable over the class of hereditarily p-spaces is not true. The most one can hope for from such a space is that it has a weaker metrizable topology. The relevant examples are the Sorgenfrey and Michael lines.

4. Cleavability over $\mathcal{S}$ and divisibility

By definition, for a space X to be cleavable over $\mathcal{S}$ means that for each $A \subseteq X$ one can find a separable metric space M_A and a continuous function $f_A : X \to M_A$ with properties as described in the introduction. Because every separable metric space embeds in the topological product space $\mathbb{R}^\omega$, we may assume that each f_A maps X into $\mathbb{R}^\omega$.

Recall that a space X is *divisible* if for each subset $A \subseteq X$ there is a countable collection $\mathcal{C}$ of open subsets of X with the property that if $x \in A$ and $y \in X - A$, then some $C \in \mathcal{C}$ has $x \in C$ and $y \notin C$. We will say the *$\mathcal{C}$ is a divisor for A*. If for any $A \subseteq X$ we can always arrange that all members of the divisor $\mathcal{C}$ for A are cozero sets of continuous real-valued functions, then we will say that X is *divisible by cozero sets*. Example **??** will show that "divisible" and "divisible by cozero sets" are different properties.

Also recall [Ar] that a space X is *weakly normal* provided for any two disjoint closed subsets A and B of X, there is a continuous $f : X \to \mathbb{R}^\omega$ with the property that $f[A] \cap f[B] = \emptyset$. The following well-known result shows that cleavability over $\mathcal{S}$ can be characterized by certain divisibility-like properties.

4.1 Proposition: *For any topological space X, the following are equivalent:*

a) *X is cleavable over $\mathcal{S}$;*

b) *X is divisible by cozero sets;*

c) *X is weakly normal and for any subset $A \subseteq X$ there is a countable collection $\mathcal{H}$ of closed sets such that if $x \in A$ and $y \in X - A$, then some $H_1,\ H_2 \in \mathcal{H}$ have $x \in H_1,\ y \in H_2$ and $H_1 \cap H_2 = \emptyset$:*

Provided X is normal, each of (a), (b), and (c) implies:

d) *for any subset* $A \subseteq X$ *there is a countable collection* $\mathcal{U}$ *of open sets such that if* $x \in A$ *and* $y \in X - A$*, then some* $U_1,\ U_2 \in \mathcal{U}$ *have* $x \in U_1,\ y \in U_2$ *and* $U_1 \cap U_2 = \emptyset$*:*

Proof. The equivalence of (a) and (c) appears in [Ar, p. 148]. To show that a) implies b), suppose that X is cleavable over $\mathcal{S}$. Given $A \subseteq X$, there is a continuous $f : X \to \mathbb{R}^\omega$ such that if $x \in A$ and $y \in X - A$, then $f(x) \neq f(y)$. Let $\mathcal{B}$ be a countable base of open sets for $\mathbb{R}^\omega$ and let $\mathcal{C} = \{f^{-1}[B] : B \in \mathcal{B}\}$. Becasue any open subset of the metric space $\mathbb{R}^\omega$ is a cozero set, each $C \in \mathcal{C}$ is a cozero set in X and so $\mathcal{C}$ is a countable collection of cozero sets as required.

To see that b) implies a), suppose that $A \subseteq X$ and suppose that $\{C_n : n \in \omega\}$ is a countable collection of cozero sets that acts as a divisor for A. For each n let $g_n : X \to \mathbb{R}$ be such that $C_n = \{x \in X : g_n(x) \neq 0\}$. Define $g : X \to \mathbb{R}^\omega$ by the rule that $g_n(x)$ is the n-th coordinate of $g(x)$ for each $x \in X$. Then g is a continuous cleaving map for A, as required.

To see that c) implies d) in case X is normal, suppose $A \subseteq X$ and $\mathcal{H}$ is as described in c). For each pair (H_1, H_2) of disjoint members of $\mathcal{H}$, choose disjoint open sets $U(H_1, H_2),\ V(H_1, H_2)$ with $H_1 \subseteq U(H_1, H_2)$ and $H_2 \subseteq V(H_1, H_2)$. Then the collection of all sets $U(H_1, H_2)$ and $V(H_1, H_2)$ is the required collection of open sets. □

4.2 Example: *There is a divisible LOTS* M^* *that is not cleavable over* $\mathcal{M}$ *and hence not cleavable over* $\mathcal{S}$*. Hence “divisible” and “divisible by cozero sets” are different properties. Furthermore,* M^* *satisfies (d) of (4.1) but does not satisfy (c), (b), or (a).*

Proof. Let M^* be the lexicographically ordered set $\{(x, n) \in \mathbb{R} \times \mathbb{Z} :$ if $x \in \mathbb{Q}$ then $n = 0\}$ with the usual open interval topology. This LOTS contains a copy of the Michael line and therefore is not metrizable. Hence it does not have a G_δ-diagonal and therefore is not cleavable over $\mathcal{M}$ or over $\mathcal{S}$ (3.1). To show that M^* is divisible and satisfies (d) of (4.1) we let $G_n = \{(x, n) : x \in \mathbb{P}\}$ and $H(p, q) = \{(x, n) \in M^* : p < x < q\}$ for each $p, q \in \mathbb{Q}$ with $p < q$. It is clear that the countable collection $\{G_n : n \in \mathbb{Z}\} \cup \{H(p, q) : p < q,\ p, q \in \mathbb{Q}\}$ separates points of M^* in the sense required by (d). The same countable collection shows that M^* is divisible. □

4.3 Proposition: *Any divisible GO-space* X *has a* σ*-disjoint open cover* $\mathcal{H}$ *that is point-separating in the sense that if* x *and* y *are distinct points of* X*, then for some* $H \in \mathcal{H},\ x \in H \subseteq X - \{y\}$*.*

Proof. Let A, B, C, D be the special subsets of X found in (2.1) and (2.2). Let $\mathcal{E} = \{A,\ B,\ C,\ D\}$. For each $E \in \mathcal{E}$ let $\mathcal{G}_E = \{G(n, E) : n \geq 1\}$ be a countable collection of open sets such that if $x \in E \subseteq X - \{y\}$, then for some n we have $x \in G(n, E) \subseteq X - \{y\}$. Let $\mathcal{H}(n, E) = \{H \subseteq X : H$ is a convex component of $G(n, E)\}$. Then $\mathcal{H} = \bigcup\{\mathcal{H}(n, E) : n \geq 1, E \in \mathcal{E}\}$ is a σ-disjoint open cover.

To complete the proof, suppose that x, y are distinct points of X. We may assume that $x < y$. There are three cases to consider.

Case 1: Suppose for some $E \in \mathcal{E}$ we have $|E \cap \{x, y\}| = 1$. Consider the case where $x \in E, y \notin E$, the other case being similar. Then for some $n,\ x \in G(n, E) \subseteq X - \{y\}$. Let H be the convex component of $G(n, E)$ containing x. Then $y \notin H$, as required.

For the rest of this proof, we will assume that Case 1 does not apply, i.e., we will assume:

(*) for each $E \in \mathcal{E},\ x \in E$ if and only if $y \in E$.

The remaining two cases depend upon the cardinality of $[x, y]$.

Case 2: Suppose that (*) holds and the set $[x, y]$ is finite. Then, using notation from Section 2, $[x, y] \subseteq N \subseteq A \cup B$. Without loss of generality, assume $x \in A$. By (*), $y \in A$. If $]x, y[\ = \emptyset$ then x and y are adjacent points, so that (2.1) yields $y \in B \subseteq X - A$, contrary to $y \in A$. Hence $]x, y[\ \neq \emptyset$ so that the immediate successor z of x lies in $]x, y[$. Then, according to (2.1), $z \notin A$ so that for some $n,\ x \in G(n, A) \subseteq X - \{z\}$. Let H be the convex component of $G(n, A)$ that contains x. If $y \in H$, then convexity of H would force $z \in [x, y] \subseteq H \subseteq G(n, A)$ contradicting $z \notin G(n, A)$. Hence $y \notin H$ as required.

Case 3: Suppose that (*) holds and $[x, y]$ is infinite. Because $X = C \cup D$ in (2.2), we may assume that $x \in C$. Choose $z \in\ ]x, y[\, \cap D$. Then $z \notin C$ so that for some $n,\ x \in G(n, C) \subseteq X - \{z\}$. Then the convex component H of $G(n, C)$ that contains x is the required member of $\mathcal{H}$ that contains x but not y. □

4.4 Corollary: *Any divisible LOTS is quasi-developable, and any perfect, divisible LOTS is metrizable.*

Proof. Let $\bigcup\{\mathcal{H}(n) : n \geq 1\}$ be the σ-disjoint open collection that separates points of X as in (4.3). Let $\mathcal{B}(m, n) = \{H_1 \cap H_2 : H_1 \in \mathcal{H}(m),\ H_2 \in \mathcal{H}(n)\}$. The collections $\mathcal{B}(m, n)$ are pairwise disjoint, and the collection $\mathcal{B} = \bigcup\{\mathcal{B}(m, n) : m, n \geq 1\}$ is a base for the LOTS X. Because X has a σ-disjoint base, X is quasi-developable [Be1]. If, in addition, X is perfect, then X is metrizable [Be2]. □

4.5 Remark: Because any GO-space with countable cellularity is perfect [L2] Corollary 4.4 strengthens a result of Kočinac [K] who proved that a divisible LOTS with countable cellularity must be metrizable. (Hence no Souslin space can be divisible.) Also note that the familiar Sorgenfrey line shows that (4.4) does not hold if we consider GO-spaces instead of linearly ordered topological spaces.

4.6 Example: *There is a metrizable linearly ordered topological space that is not divisible. This space is a LOTS that is cleavable over $\mathcal{M}$ but is not cleavable over $\mathcal{S}$.*

Proof. Let T be any dense in itself subset of $\mathbb{R}$ having no endpoints and having $2^{|T|} > c$. For each $\alpha < c^+$ write $T_\alpha = \{\alpha\} \times T$ and order the set $X = \bigcup\{T_\alpha : \alpha < \mathbf{c}^+\}$ lexicographically. Choose a family $\{S_\alpha : \alpha < c^+\}$ of distinct subsets of T. Let $A = \bigcup\{\{\alpha\} \times S_\alpha : \alpha < c^+\}$. If X were divisible, there would be a sequence G_n of open subsets of X such that if $(\alpha, x) \in A \subseteq X - \{(\beta, y)\}$ then some G_n has $(\alpha, x) \in G_n \subseteq X - \{(\beta, y)\}$. Let $\pi : X \to T$ be given by $\pi(\alpha, x) = x$ and for each $\alpha < c^+$ let $\sigma_\alpha = \langle \pi[G_1 \cap T_\alpha], \pi[G_2 \cap T_\alpha], \pi[G_3 \cap T_\alpha], \cdots \rangle$. Each σ_α is a sequence of open subsets of T, and the collection of all sequences of open subsets of the separable metric space T has cardinality $\mathbf{c}$. Hence there are distinct indices $\alpha, \beta < \mathbf{c}^+$ such that $\sigma_\alpha = \sigma_\beta$. But $S_\alpha \neq S_\beta$. Without loss of generality we may choose $x \in S_\alpha - S_\beta$. Then $(\alpha, x) \in A$ and $(\beta, x) \in X - A$ so that for some n, $(\alpha, x) \in G_n \subseteq X - \{(\beta, x)\}$. Then $x \in \pi[G_n \cap T_\alpha]$ and $x \notin \pi[G_n \cap T_\beta]$ and that is impossible because $\sigma_\alpha = \sigma_\beta$. Therefore X is not divisible, as claimed. Observe that by taking $T = \mathbb{R}$ or $T = \mathbb{P}$ we obtain (respectively) an example that is either locally compact or Čech-complete and zero dimensional. □

It is reasonable to ask whether the characterization theorem for cleavability over $\mathcal{M}$ has an analog for cleavability over $\mathcal{S}$. Based upon experience in metrization theory, one might wonder whether one could simply replace "σ-discrete" by "countable" to obtain the desired result. Our next result (4.7) gives such a characterization theorem for GO-spaces with cellularity at most c, and a subsequent example (4.9) shows that some restriction on weight, cellularity, or cardinality is needed for (4.7) to hold.

4.7 Theorem: *Let $(X, \mathcal{T}, <)$ be a GO-space. Then the following are equivalent:*

a) *there is a continuous 1-1 mapping from X into $\mathbb{R}^\omega$, i.e., X is absolutely cleavable over $\mathcal{S}$;*

b) there is a separable metric topology $\mathcal{T}_m$ on X having $\mathcal{T}_m \subseteq \mathcal{T}$;

c) $(X, \mathcal{T})$ is cleavable over $\mathcal{S}$ and $c(X) \leq \mathbf{c}$;

d) $(X, \mathcal{T})$ is divisible by cozero sets and $c(X) \leq \mathbf{c}$;

e) $(X, \mathcal{T})$ has a countable, point-separating cover by cozero sets.

Furthermore, $|X| \leq \mathbf{c}$ for any GO-space satisfying one of (a) through (e).

Proof. Assertions (a) and (e) are equivalent for any topological space (see 2.3-b) as are assertions (c) and (d) (see 4.1) and assertions (a) and (b). Next suppose that a) holds. Then X is cleavable over $\mathcal{S}$, and if $c(X) > \mathbf{c}$, then $|X| > \mathbf{c}$ which is impossible, given (a). Thus, a) implies c). Therefore it will be enough to prove (c) implies (a).

To prove that (c) implies (a), suppose that X is a GO-space with cellularity at most $\mathbf{c}$ and suppose X is cleavable over $\mathcal{S}$. Because $\mathcal{S} \subseteq \mathcal{M}$, (3.1) yields a metrizable topology $\mathcal{T}_m$ having $\mathcal{T}_m \subseteq \mathcal{T}$. Then $c(X, \mathcal{T}_m) \leq \mathbf{c}$ so that $w(X, \mathcal{T}_m) \leq \mathbf{c}$. According to Kowalsky's theorem [E, 4.4.9] there is a homeomorphism h from $(X, \mathcal{T}_m)$ onto a subspace of H^ω where H is the metric hedgehog with $\mathbf{c}$ spines. Then the function h is a 1-1 continuous function from $(X, \mathcal{T})$ into H^ω. But, as noted in [ArS], there is a 1-1 continuous mapping from H into the Euclidean space $\mathbb{R}^2$ and hence there is a 1-1 continuous mapping g from H^ω into $(\mathbb{R}^2)^\omega$. The composite mapping $g \circ h$ is a continuous 1-1 mapping from $(X, \mathcal{T})$ into the separable metric space $(\mathbb{R}^2)^\omega$, as required by (a). □

4.8 Example: *For each $\kappa > \mathbf{c}$, there is a dense-in-itself metrizable LOTS X that is cleavable over $\mathcal{S}$ and has $|X| = \kappa$. Hence the cellularity restriction in (4.7) is necessary.*

Proof. Let $\kappa > \mathbf{c}$ and let X be the lexicographic product $[0, \kappa[\times \mathbb{Q}$. We will show in (5.11) that X is cleavable over $\mathbb{Q}$ and hence is cleavable over $\mathcal{S}$. But X cannot have a countable point-separating open cover, or a weaker separable metric topology, because each of those conditions implies $|X| \leq \mathbf{c}$. □

4.9 Remark: It is tempting to suppose that if the space X of (4.7) is a LOTS with $c(X) \leq \mathbf{c}$, then from (4.7-e) one could get a countable base for X by taking pairwise intersections of the countable, point-separating open cover $\mathcal{C}$, as in the proof of (4.4). However, Example 4.8 shows that will not be the case. The argument in (4.4) does not apply here because the sets in the countable point-separating open cover $\mathcal{C}$

are not necessarily convex, so that taking pairwise intersections does not yield a base for X.

4.10 Open Problems: *(a) Characterize GO-spaces that are cleavable over $\mathcal{S}$. (b) Characterize GO-spaces that are divisible.* Note that a solution for part (b) would combine with (4.1) to solve part (a).

5. Cleavability over $\mathbb{R}, \mathbb{P}$, and $\mathbb{Q}$

Recall that $\mathbb{R}$, $\mathbb{P}$, and $\mathbb{Q}$ denote, respectively, the usual sets of real, irrational, and rational numbers. We begin with an example showing that there is a metrizable LOTS that is cleavable over $\mathcal{S}$ but not over $\mathbb{R}$ (and hence not cleavable over $\mathbb{P}$ or $\mathbb{Q}$ either). That result needs a lemma that might be of independent interest.

5.1 Lemma: *There is a collection $\{A(\alpha) : \alpha < \mathbf{c}\}$ of subsets of $\mathbb{R}$ satisfying:*

a) *for each non-empty open subset $V \subseteq \mathbb{R}$ and for each $\alpha < \mathbf{c}$, both $V \cap A(\alpha)$ and $V - A(\alpha)$ have cardinality $\mathbf{c}$;*

b) *whenever $0 \leq \alpha < \beta < \mathbf{c}$ and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f[\mathbb{R}] \cap g[\mathbb{R}]$ is non-degenerate (i.e. has more than a single point), then there are points $x \in A(\alpha)$ and $x' \in \mathbb{R} - A(\beta)$ with $f(x) = g(x')$.*

Proof. Let $\mathcal{C} = \{\langle f, g, \alpha, \beta\rangle : 0 \leq \alpha, \beta < \mathbf{c}$, and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, and $|f[\mathbb{R}] \cap g[\mathbb{R}]| > 1\}$. Then $|\mathcal{C}| = \mathbf{c}$ so we may index $\mathcal{C}$ as $\{\langle f_\gamma, g_\gamma, \alpha_\gamma, \beta_\gamma\rangle : 0 \leq \gamma < \mathbf{c}\}$. Next, let $\mathbb{R} = \{r_\eta : \eta < \mathbf{c}\}$ be any well-ordering of $\mathbb{R}$ and let $\{V_n : n \in \omega\}$ be any countable base for $\mathbb{R}$ with $V_0 = \mathbb{R}$. For each $\alpha < \mathbf{c}$ we will define sets of real numbers $A(\alpha, \delta)$, $C(\alpha, \delta)$ and $U(\delta)$ by recursion on δ in such a way that:

1) $A(\alpha, \delta) \cap C(\alpha, \delta) = \emptyset$;

2) $A(\alpha, \delta) \cup C(\alpha, \delta) = U(\delta)$;

3) $|U(\delta)| < \mathbf{c}$ and $|U(\delta + 1) - U(\delta)| < \omega$;

4) if $0 \leq \eta_1 < \eta_2 \leq \delta$ then $A(\alpha, \eta_1) \subset A(\alpha, \eta_2)$ and $C(\alpha, \eta_1) \subseteq C(\alpha, \eta_2)$;

5) if δ is a limit ordinal, then $\{r_\eta : \eta < \delta\} \subseteq U(\delta)$.

We will use the sets $A(\alpha, \delta)$ and $C(\alpha, \delta)$ to build, respectively, the sets $A(\alpha)$ and $\mathbb{R} - A(\alpha)$, and the set $U(\delta)$ will be the set of points used in the construction up to and including stage δ.

We initialize the recursion by letting $U(0) = A(\alpha, 0) = C(\alpha, 0) = \emptyset$ for every α. Now suppose that $\gamma < \mathbf{c}$ and that we have chosen sets $A(\alpha, \delta), C(\alpha, \delta)$ and $U(\delta)$ for each $\delta < \gamma$ in such a way that (1) through (5) are satisfied for each $\alpha < \mathbf{c}$. Define $A^*(\alpha, \gamma) = \bigcup\{A(\alpha, \delta) : \delta < \gamma\}$, $C^*(\alpha, \gamma) = \bigcup\{C(\alpha, \delta) : \delta < \gamma\}$ and $U^*(\gamma) = \bigcup\{U(\delta) : \delta < \gamma\}$. Because only finitely many points are added at each stage, $|U^*(\gamma)| < \mathbf{c}$.

Because $f_\gamma[\mathbb{R}] \cap g_\gamma[\mathbb{R}]$ is a non-degenerate convex subset of $\mathbb{R}$, we know that $f_\gamma[\mathbb{R}] \cap g_\gamma[\mathbb{R}]$ has cardinality c, so there is a point $z_\gamma \in (f_\gamma[\mathbb{R}] \cap g_\gamma[\mathbb{R}]) - (f_\gamma[U^*(\gamma)] \cup g_\gamma[U^*(\gamma)] \cup U^*(\gamma))$ and hence points $x_\gamma,\ x'_\gamma \in \mathbb{R} - U^*(\gamma)$ with $f_\gamma(x_\gamma) = z_\gamma = g_\gamma(x'_\gamma)$. (We do not know whether x_γ and x'_γ are distinct, and that will lead to two cases below.)

There is a unique λ and n such that $\gamma = \lambda + n$ where $\lambda = 0$ or λ is a limit ordinal, and where $0 \leq n < \omega$. Let y_γ and y'_γ be the two points of $V_n - (U^*(\gamma) \cup \{x_\gamma, x'_\gamma\})$ having the least possible indices in the fixed well-ordering $\{r_\eta : \eta < \mathbf{c}\}$. Note that none of the points $x_\gamma, x'_\gamma, y_\gamma, y'_\gamma$ were chosen at any earlier stage of the construction. We now define $U(\gamma) = U^*(\gamma) \cup \{x_\gamma, x'_\gamma, y_\gamma, y'_\gamma\}$ and we define the sets $A(\alpha, \gamma)$ and $C(\alpha, \gamma)$ by the rules:

a) if $\alpha \notin \{\alpha_\gamma, \beta_\gamma\}$ then $A(\alpha, \gamma) = A^*(\alpha, \gamma) \cup \{x_\gamma, x'_\gamma, y_\gamma\}$ and $C(\alpha, \gamma) = C^*(\alpha, \gamma) \cup \{y'_\gamma\}$;

The next two cases complete the recursion by defining the sets $A(\alpha_\gamma, \gamma)$, $C(\alpha_\gamma, \gamma)$, $A(\beta_\gamma, \gamma)$ and $C(\beta_\gamma, \gamma)$.

b) if $x_\gamma = x'_\gamma$ then

$$\begin{aligned}
A(\alpha_\gamma, \gamma) &= A^*(\alpha_\gamma, \gamma) \cup \{x_\gamma, y_\gamma\};\\
C(\alpha_\gamma, \gamma) &= C^*(\alpha_\gamma, \gamma) \cup \{y'_\gamma\};\\
A(\beta_\gamma, \gamma) &= A^*(\beta_\gamma, \gamma) \cup \{y_\gamma\};\\
C(\beta_\gamma, \gamma) &= C^*(\beta_\gamma, \gamma) \cup \{x'_\gamma, y'_\gamma\}.
\end{aligned}$$

c) If $x_\gamma \neq x'_\gamma$ then:

$$\begin{aligned}
A(\alpha_\gamma, \gamma) &= A^*(\alpha_\gamma, \gamma) \cup \{x_\gamma, y_\gamma\};\\
C(\alpha_\gamma, \gamma) &= C^*(\alpha_\gamma, \gamma) \cup \{x'_\gamma, y'_\gamma\};\\
A(\beta_\gamma, \gamma) &= A^*(\beta_\gamma, \gamma) \cup \{x_\gamma, y_\gamma)\};\\
C(\beta_\gamma, \gamma) &= C^*(\beta_\gamma, \gamma) \cup \{x'_\gamma, y'_\gamma\}.
\end{aligned}$$

It is clear that the definitions above yield sets that satisfy (1), (2), (3), and (4). To verify that (5) is satisfied for γ, suppose γ is a limit ordinal and $\{r_\eta : \eta < \gamma\} \not\subseteq U(\gamma)$. Let η be the least ordinal with $\eta < \gamma$ and $r_\eta \notin U(\gamma)$. If there is a limit ordinal λ with $\eta < \lambda < \gamma$, then the induction hypothesis would show that $r_\eta \in U(\lambda) \subseteq U(\gamma)$, contrary to our choice of η. Hence γ has the form $\lambda + \omega$ where λ is a limit ordinal, or zero, and $\lambda \leq \eta$. Because $r_\eta \notin U(\gamma)$ we certainly have $r_\eta \notin U^*(\gamma)$. Hence either $r_\eta \in \{x_\gamma, x'_\gamma\}$ or else r_η is the first point of $\mathbb{R} - (U^*(\gamma) \cup \{x_\gamma, x'_\gamma\}) = V_0 - (U^*(\gamma) \cup \{x_\gamma, x'_\gamma\})$, and in the latter case $r_\eta \in \{y_\gamma, y'_\gamma\}$. Hence $r_\eta \in U(\gamma)$ contrary to our choice of η. Thus (5) holds and the recursion continues.

Given the sets $A(\alpha,\gamma), C(\alpha,\gamma)$ and $U(\gamma)$ satisfying (1) - (5) for every $\alpha, \gamma < \mathbf{c}$, we see that $\mathbb{R} = \bigcup\{U(\gamma) : \gamma < \mathbf{c}\}$. Define $A(\alpha) = \bigcup\{A(\alpha,\gamma) : \gamma < \mathbf{c}\}$ and note that $\mathbb{R} - A(\alpha) = \bigcup\{C(\alpha,\gamma) : \gamma < \mathbf{c}\}$.

Assertion (a) of the lemma follows from the fact that if V is a nonempty open subset of $\mathbb{R}$, then for some n, $V_n \subseteq V$ so that $\mathbf{c}$ many points of $A(\alpha)$ and of $\mathbb{R} - A(\alpha)$ are chosen from V_n. To verify assertion (b) of the lemma, suppose $\alpha \neq \beta$ and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f[\mathbb{R}] \cap g[\mathbb{R}]$ is non-degenerate. Then for some $\gamma < \mathbf{c}$ we have $\langle f, g, \alpha, \beta\rangle = \langle f_\gamma, g_\gamma, \alpha_\gamma, \beta_\gamma\rangle$ so that the points x_γ and x'_γ have $x_\gamma \in A(\alpha,\gamma) \subseteq A(\alpha)$ and $x'_\gamma \in C(\alpha,\gamma) \subseteq \mathbb{R} - A(\alpha)$, and $f(x_\gamma) = g(x'_\gamma)$, as required. □

5.2 Example: *The lexicographic product space* $X = [0, \omega_1[\times \mathbb{R}$ *is a metrizable LOTS that is absolutely cleavable over* $\mathcal{S}$ *but is not cleavable over* $\mathbb{R}$.

Proof. Let $S \subset \mathbb{R}$ have $|S| = \omega_1$. Then X is homeomorphic to the lexicographic product $Y = S \times \mathbb{R}$ and the identity function from Y into $\mathbb{R}^2$ is continuous and 1-1. Hence X is absolutely cleavable over $\mathcal{S}$. To see that X is not cleavable over $\mathbb{R}$, we choose sets $\{A(\alpha) : 0 \leq \alpha < \omega_1\}$ from among the sets constructed in (5.1). We let $S = \bigcup\{\{\alpha\} \times A(\alpha) : \alpha < \omega_1\}$ and we write $V_\alpha = \{\alpha\} \times \mathbb{R}$.

Suppose $\psi : X \to \mathbb{R}$ is any continuous function. We will show that there is a point $p \in S$ and a point $q \in X - S$ with $\psi(p) = \psi(q)$, and that will show that X is not cleavable over $\mathbb{R}$. Define $f_\alpha(x) = \psi(\alpha, x)$ for each $x \in \mathbb{R}$. Each f_α is a continuous function. If there is some α such that the set $f_\alpha[\mathbb{R}]$ is a singleton, then we choose any $x \in A(\alpha)$ and any $y \in R - A(\alpha)$ and let $p = (\alpha, x)$ and $q = (\alpha, y)$ so that $\psi(p) = \psi(q)$ as required. So assume that each set $f_\alpha[\mathbb{R}]$ is non-degenerate. Because $\mathbb{R}$ is separable, we can find $0 \leq \alpha < \beta < \omega_1$ such that $f_\alpha[\mathbb{R}] \cap f_\beta[\mathbb{R}]$ is non-degenerate. Consequently, there are points $x \in A(\alpha)$ and $x' \in$

$\mathbb{R} - A(\beta)$ with $f_\alpha(x) = f_\beta(x')$. Letting $p = (\alpha, x)$ and $q = (\beta, x')$, we have $p \in S$, $q \in X - S$ and $\psi(p) = \psi(q)$, as required. Hence X is not cleavable over $\mathbb{R}$. $\square$

5.3 Open Problem: *Characterize GO-spaces that are cleavable over* $\mathbb{R}$.

We know a little about Problem 5.3. First, there are no cardinality restrictions on a LOTS that is cleavable over $\mathbb{R}$, because any discrete space is cleavable over $\mathbb{R}$, and can be ordered in such a way that it is a LOTS. Second, a GO-space that is cleavable over $\mathbb{R}$ can have at most countably many non-degenerate connected components. (Proof: If X is a GO-space that is cleavable over $\mathbb{R}$, then each non-degenerate connected component C of X is homeomorphic to a connected subspace of $\mathbb{R}$. Removing at most two points from such a C gives us a space C' that is homeomorphic to $\mathbb{R}$. Let $Y = \bigcup\{C' : C$ is a non-degenerate connected component of $X\}$. Because X is cleavable over $\mathbb{R}$, so is Y. Therefore, for X to have more than countably many non-degenerate connected components would contradict (5.2).) Further information about GO-spaces that are cleavable over $\mathbb{R}$ comes from Example (4.6) which shows that such a GO-space cannot contain more that $\mathbf{c}$ pairwise disjoint open sets each of which contains a topological copy of a fixed subspace T of $\mathbb{R}$, where $2^{|T|} > \mathbf{c}$. $\square$

Characterizing GO-spaces that are cleavable over $\mathbb{P}$ is an easier problem. Recall that $ind(X) = 0$ means that each point of X has a base of clopen sets and that $Ind(X) = 0$ means that whenever A is closed and U is open and $A \subseteq U$, then there is a clopen set V with $A \subseteq V \subseteq U$. In general, these dimension functions may be different, but for GO-spaces we have the following well-known fact:

5.4 Lemma: *For a GO-space X, the following are equivalent:*

a) *$ind(X) = 0$;*

b) *$Ind(X) = 0$;*

c) *no connected subset of X has more than one point.*

5.5 Theorem: *The following are equivalent for a GO-space X:*

a) *X is cleavable over $\mathbb{P}$;*

b) *X is cleavable over $\mathbb{R}$ and $ind(X) = 0$;*

c) *X is cleavable over $\mathcal{S}$ and $ind(X) = 0$;*

d) For every subset $A \subset X$, there is a countable collection $\mathcal{C}$ of clopen subsets of X such that if $x \in A \subseteq X - \{y\}$, then some $C \in \mathcal{C}$ has $x \in C \subseteq X - \{y\}$.

Proof. Clearly b) implies c). We will show that a) implies b), c) implies d), and d) implies a).

To show that a) implies b), suppose that X satisfies a). Being cleavable over $\mathbb{P}$, X is also cleavable over $\mathbb{R}$. To show that $ind(X) = 0$, in the light of (5.4) it will be enough to show that if $C \subseteq X$ is connected, then $|C| \leq 1$. For contradiction, suppose there are distinct points x, y of C. Then $f[C]$ is a connected subset of P containing both $f(x)$ and $f(y)$ whenever $f : X \to P$ is continuous, so that $f(x) = f(y)$. Thus there is no cleaving function for the set $A = \{x\}$. Thus a) implies b).

Next we show that c) implies d). Let $A \subseteq X$. Using c), choose a continuous $f : X \to \mathbb{R}^\omega$ that is a cleaving function for A. Let $\{B_n : n \geq 1\}$ be a countable base of open sets for $\mathbb{R}^\omega$. Then each set $f^{-1}[B_n]$ is an open F_σ-subset of X. Because $Ind(X) = 0$ (see 5.4), each $f^{-1}[B_n]$ can be written as $\bigcup\{C(n,k) : k \geq 1\}$ where each $C(n,k)$ is a clopen subset of X. Then $\{C(n,k) : n, k \geq 1\}$ is the required clopen collection, so c) implies d)

Finally we show that d) implies a). Let $A \subseteq X$ and let $\mathcal{C} = \{C_n : n \in \omega\}$ be the clopen collection given by (d). For each n, let $f_n : X \to \mathbb{Z}$ be the characteristic function of the set C_n. Define $f : X \to \mathbb{Z}^\omega$ by $f(x) = \langle f_0(x), f_1(x), f_2(x), \cdots \rangle$. The f is continuous and since $\mathbb{Z}^\omega$ is homeomorphic to $\mathbb{P}$, f is the desired cleaving function for A. Thus (d) implies (a). $\square$

Proposition 2.3 characterizes topological spaces that can be mapped into $\mathbb{P}$ by a 1-1 continuous function (i.e., are absolutely cleavable over $\mathbb{P}$). Example 5.6 will show that for GO-spaces, this class is strictly smaller than the class of GO-spaces that are cleavable over $\mathbb{P}$.

5.6 Example: *(a) There is a dense-in-itself, metrizable LOTS that is cleavable over $\mathbb{P}$ but not absolutely cleavable over $\mathbb{P}$. (b) The lexicographic product space $Y_\kappa = [0, \kappa[\times \mathbb{P}$ is cleavable over $\mathbb{R}$ if and only if $\kappa \leq \mathbf{c}$.*

Proof. To obtain the example announced in (a), we invoke (5.10), below, to show that the lexicographically ordered LOTS $X = [0, c^+[\times \mathbb{Q}$ is cleavable over $\mathbb{Q}$ and hence also over $\mathbb{P}$. But the cardinality of X is too large for there to be a 1-1 function from X into $\mathbb{P}$. To obtain (b), we apply (4.6) with $T = \mathbb{P}$ to show that if $\kappa > \mathbf{c}$ then Y_κ is not divisible and hence not cleavable over $\mathbb{R}$. In case $\kappa \leq \mathbf{c}$ then we can replace the set $[0,\kappa[$ by a subset $S \subset \mathbb{R}$ of cardinality κ and then we see a natural 1-1 continuous mapping from Y_κ into $\mathbb{R}^2$. Thus Y_κ is cleavable over $\mathcal{S}$. Now apply the equivalence of (a) and (c) in (5.5) to conclude that Y_κ is cleavable over $\mathbb{P}$ and hence over $\mathbb{R}$. □

As an application of (2.3) we can show that a certain strange metrizable space constructed by A.H. Stone in [St] is absolutely cleavable over $\mathbb{P}$.

5.7 Example: *Stone's metric space X is a LOTS satisfying:*

a) $|X| = \omega_1$;

b) X is not the union of countably many discrete subspaces;

c) any separable subspace of X is countable;

d) there is a continuous 1-1 mapping from X into $\mathbb{P}$.

Proof. That Stone's metric space X satisfies a), b), and c) appears in [St]. That X is a LOTS under some ordering follows from Herrlich's orderability theorem ([Hr], see also [E, 2.3.6]). Because X is constructed as a certain subspace of D^ω where D is a discrete space of cardinality ω_1, X has a σ-discrete clopen cover that separates points as in (2.3-c). Hence, by (2.3) there is a continuous 1-1 mapping from X into $\mathbb{P}$. □

The final result in this section characterizes GO-spaces that are cleavable over $\mathbb{Q}$. It is a corollary of a more general result given by Arhangel'skii in [Ar].

5.8 Theorem: ([Ar]) *: Suppose that $Ind(X) = 0$ and that each subset of the topological space X is an F_σ-subset of X. Then X is cleavable over $\mathbb{Q}$.*

5.9 Corollary: *Let X be a GO-space. Then X is cleavable over $\mathbb{Q}$ if and only if each subset of X is an F_σ-subset of X.*

Proof. The implication $\Rightarrow$ is clear for any topological space. The converse follows from (5.8) and (5.4) once we show that there are no non-degenerate connected sets in a GO-space X in which each subset is an F_σ-set. So suppose that C is a connected subset of X with more than one point. As a subspace of X, the space C will be a connected, non-degenerate LOTS in which each singleton is a G_δ-set. Thus C is first countable. Furthermore, C has no isolated points so that, by a theorem of Hewitt [Hw], the space C is resolvable, i.e., contains two disjoint dense sets D_1, D_2. Replacing D_2 by $C - D_1$ if necessary, we may assume that $D_1 \cup D_2 = C$. Each D_i is of first category in C, being a dense and co-dense F_σ-set. But then C is first category in itself, and that is impossible because C is a connected LOTS (and therefore locally compact) so that C satisfies the Baire Category Theorem. Therefore, connected subsets of X must be degenerate. □

5.10 Corollary: *Any locally countable metric space is cleavable over* $\mathbb{Q}$.

Proof. Any locally countable metric space X is a topological sum of countable metric spaces, i.e., of subspaces of $\mathbb{Q}$. Thus X embeds in a topological sum of copies of $\mathbb{Q}$. But any topological sum of copies of $\mathbb{Q}$ is homeomorphic to the lexicographically ordered LOTS $[0, \kappa[\times \mathbb{Q}$ for suitably large κ, and by (5.8) such a LOTS is cleavable over $\mathbb{Q}$. Hence so is its subspace X. □

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