The Monotone Lindelöf Property and Separability in Ordered Spaces

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Abstract

In this paper we investigate the relation between separability and the monotone Lindelöf property in generalized ordered (GO)-spaces. We examine which classical examples are or are not monotonically Lindelöf. Using a new technique for investigating open covers of GO-spaces, we show that any separable GO-space is hereditarily monotonically Lindelöf. Finally, we investigate the relation between the hereditarily monotonically Lindelöf property and the Souslin problem.

Key words and phrases: monotone Lindelöf property, GO-space, generalized ordered space, separability

Math Reviews classifications (2000): Primary 54F05; Secondary 54D20, 54D65

*****draft of May 29, 2003*****

1 Introduction

A topological space X is *monotonically Lindelöf* [2] if for each open cover $\mathcal{U}$ of X there is a countable open cover $r(\mathcal{U})$ of X that refines $\mathcal{U}$ and has the property that if an open cover $\mathcal{U}$ refines an open cover $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. In this case, r will be called a *monotone Lindelöf operator* for the space X. The role of the monotone Lindelöf property in the theory of generalized ordered spaces is far from clear, and this paper we present the basic results and pose a family of open questions.

We show that the compact LOTS $[0, \omega_1]$ is not monotonically Lindelöf and deduce that a compact, monotonically Lindelöf LOTS is first countable. We show that there is a monotonically Lindelöf LOTS that is not first countable, that there is a compact monotonically Lindelöf LOTS that is not perfect (i.e., some closed set is not a G_δ-set). We prove that the branch space of certain Aronszajn trees must be monotonically Lindelöf and that the branch space of certain Souslin trees must be hereditarily monotonically Lindelöf. Finally, we show that any separable GO-space is monotonically Lindelöf and discuss the relation between the Souslin problem and the question of whether separability is equivalent to the hereditary monotone Lindelöf property in GO-spaces. Several of these results depend on a new technique for investigating open covers of GO-spaces.

Recall that a *generalized ordered space* (GO-space) is a triple $(X, \mathcal{T}, <)$ where $<$ is a linear ordering of X and $\mathcal{T}$ is a Hausdorff topology on X that has a base of order-convex sets. If $\mathcal{T}$ is the open interval topology of the given ordering, then $(X, \mathcal{T}, <)$ is a *linearly ordered topological space* (LOTS). It is known that the generalized ordered spaces are exactly those spaces that can be homeomorphically embedded in some LOTS.

In this paper, we reserve the symbols $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ for the sets of all integers, rational numbers, and real numbers, respectively. For a set S and a collection $\mathcal{U}$ of sets, we will write $S \prec \mathcal{U}$ to mean that S is a subset of some member of $\mathcal{U}$.

2 Classical Examples and the Monotone Lindelöf Property

In this section, we investigate the monotone Lindelöf property in several familiar ordered spaces – $[0, \omega_1]$, the lexicographic square, and a branch space of an ω-branching Aronszajn tree.

Lemma 2.1 *Any separable metric space is monotonically Lindelöf.*

Proof: Let $\mathcal{B}$ be a countable base for the space X. For any open cover $\mathcal{U}$ of X, let $r(\mathcal{U}) = \{B \in \mathcal{B} : \exists U \in \mathcal{U}\ with\ B \subseteq U\}$ $\Box$

There exist countable regular spaces that are not monotonically Lindelöf [3], but (of course) these examples are not GO-spaces.

Example 2.2 *There is a monotonically Lindelöf LOTS that is not first countable.*

Proof: Let X be the lexicographically ordered LOTS given by

$$X = ([0, \omega_1) \times \mathbb{Z}) \cup \{(\omega_1, 0)\}.$$

For any open cover $\mathcal{U}$ of X, there is a first ordinal $\alpha = \alpha(\mathcal{U})$ such that the interval $((\alpha, 0), (\omega_1, 0)]$ is a subset of some member of $\mathcal{U}$. Define $r(\mathcal{U}) = \{((\alpha, 0), (\omega_1, 0)]\} \cup \{\{(\beta, k)\} : \beta < \alpha\ and\ k \in \mathbb{Z}\ or\ \beta = \alpha\ and\ k < 0\}$. $\Box$

The space of Example 2.2 is not *hereditarily* monotonically Lindelöf because it is not even hereditarily Lindelöf. GO-spaces that are hereditarily monotonically Lindelöf will be studied in the next section.

Example 2.3 *The ordinal space $[0, \omega_1]$ is a compact LOTS that is not monotonically Lindelöf.*

Proof: For contradiction, suppose that r is a monotone Lindelöf operator for $[0, \omega_1]$. For each $\alpha < \omega_1$ let

$$\mathcal{U}_\alpha = \{[0, \beta) : \beta < \omega_1\} \cup \{(\alpha, \omega_1]\}.$$

Note that if $\alpha < \beta$ then $\mathcal{U}_\beta$ refines $\mathcal{U}_\alpha$ so that $r(\mathcal{U}_\beta)$ refines $r(\mathcal{U}_\alpha)$.

For each $\alpha < \omega_1$ some members of $r(\mathcal{U}_\alpha)$ will be countable, while others will not. Define

$$\beta(\alpha) = \sup\left(\bigcup\{V \in r(\mathcal{U}_\alpha) : |V| \leq \omega\}\right) + 1$$

and note that $\beta(\alpha) < \omega_1$.

Claim 1: We claim that $\alpha < \beta(\alpha)$. Choose any $V_1 \in r(\mathcal{U}_\alpha)$ with $\alpha \in V_1$. Choose $U_1 \in \mathcal{U}_\alpha$ with $V_1 \subseteq U_1$. Then either $U_1 = (\alpha, \omega_1]$ or else U_1 is countable. Because $\alpha \in U_1$, the first option is impossible, so U_1, and hence also V_1, is countable. Therefore $\alpha \in V_1 \subseteq [0, \beta(\alpha))$, as claimed.

Claim 2: If $\alpha < \gamma$ and $V \in r(\mathcal{U}_\gamma)$, then either $V \subseteq [0, \beta(\alpha))$ or else $V \subseteq (\alpha, \omega_1]$. We know that $r(\mathcal{U}_\gamma)$ refines $r(\mathcal{U}_\alpha)$, so we may choose some $W \in r(\mathcal{U}_\alpha)$ and some $U \in \mathcal{U}_\alpha$ with $V \subseteq W \subseteq U$. If U is countable, then so is W and then $V \subseteq W \subseteq [0, \beta(\alpha))$. If U is not countable, then $U = (\alpha, \omega_1]$ so that $V \subseteq (\alpha, \omega_1]$, as claimed.

Let $\alpha_0 = \omega$ and choose $\alpha_1 = \beta(\alpha_0) + 1$. Recursively define α_n so that $\alpha_{n+1} = \beta(\alpha_n) + 1$. Let $\gamma = \sup\{\alpha_n : n < \omega\}$. Then γ is a limit ordinal.

Claim 3: If $\alpha_n \in V \in \mathcal{U}_\gamma$, then $\alpha_{n+1} \notin V$. Note that $\alpha_n < \gamma$ and apply Claim 2 to conclude that either $V \subseteq [0, \beta(\alpha_n))$ or else $V \subseteq (\alpha_n, \omega_1]$. The second option is impossible because $\alpha_n \in V$, so that $V \subseteq [0, \beta(\alpha_n))$. But $\alpha_{n+1} > \beta(\alpha_n)$ so that $\alpha_{n+1} \notin V$, as claimed.

To complete the proof, choose any $V \in r(\mathcal{U}_\gamma)$ that contains γ. Because $\gamma = \sup\{\alpha_k : k < \omega\}$ there must be some n such that both α_n and α_{n+1} belong to V, contradicting Claim 3. □

In Example 2.2 we described a monotonically Lindelöf LOTS that is not first countable. By way of contrast, an immediate corollary of Example 2.3 is:

Corollary 2.4 *Any monotonically Lindelöf compact LOTS is first countable.*

Proof: If a compact LOTS X is not first countable, it contains a closed subset Y that is homeomorphic to $[0, \omega_1]$. Being closed in X, the subspace Y inherits the monotone Lindelöf property, and that is impossible in the light of Example 2.3. □

Example 2.2 shows that the previous corollary does not hold for GO-spaces that are not compact.

Example 2.5 *The lexicographic square* $[0, 1] \times [0, 1]$ *is compact and monotonically Lindelöf but is not perfect.*

Proof: Parts of this proof closely parallel the detailed proof of Proposition 3.1 in the next section. Consequently, some details of this proof are omitted.

It is well-known that the lexicographic square is compact and not perfect. To verify the monotonic Lindelöf property, it will be notationaly convenient to show that the lexicographic rectangle $X = \mathbb{R} \times [0, 1]$ is monotonically Lindelöf. Then so is its closed subspace $[0, 1] \times [0, 1]$.

Let $E = \mathbb{Q} \times \{0\}$ and for each $x \in \mathbb{R}$, let r_x be a monotonic Lindelöf operator for the separable metric subspace $V(x) = \{x\} \times (0, 1)$ of X (see 2.1).

For any open cover $\mathcal{U}$ of X, let $r_1(\mathcal{U}) = \{(e_1, e_2) : e_i \in E,\ e_1 < e_2,\ and\ (e_1, e_2) \prec \mathcal{U}\}$. Observe that if $(x, t) \in X$ and some member of $\mathcal{U}$ contains both points $(x, 0)$ and $(x, 1)$, then (x, t) is covered by $r_1(\mathcal{U})$. Let $S(\mathcal{U}) = \{x \in \mathbb{R} :\ no\ member\ of\ \mathcal{U}\ contains\ both\ (x, 0)\ and\ (x, 1)\}$. Because any uncountable subset of $\mathbb{R}$ contains a limit point of itself, one can show that the set S is countable. Let $r_2(\mathcal{U}) = \bigcup\{r_x(\mathcal{U}|_{V(x)}) : x \in S(\mathcal{U})\}$. Then $r_2(\mathcal{U})$ is countable, and $r_1(\mathcal{U}) \cup r_2(\mathcal{U})$ covers all of X except for certain points (x, i) with $i \in 0, 1$.

For each $x \in \mathbb{R}$, let

$$RF(x,\mathcal{U}) = \{e \in E : (x, 1) < e\ and\ \exists m \geq 1\ with\ ((x, 1 - \frac{1}{m}), e) \prec \mathcal{U}\},$$

$$RG(x,\mathcal{U}) = \{e \in E : (x, 1) < e\ and\ \exists y < x\ with\ ((y, 0), e) \prec \mathcal{U}\}.$$

Then $RG(x,\mathcal{U}) \subseteq RF(x,\mathcal{U})$. Let $RD(\mathcal{U}) = \{x \in \mathbb{R} : RG(x,\mathcal{U}) \neq RF(x,\mathcal{U})\}$. Let

$$r_3(\mathcal{U}) = \{((x, 1 - \frac{1}{m}), e) : x \in RD(\mathcal{U}),\ e \in RF(x,\mathcal{U}),\ m \geq 1,\ and\ ((x, 1 - \frac{1}{m}), e) \prec \mathcal{U}\}.$$

Then $r_3(\mathcal{U})$ is countable.

For each $x \in \mathbb{R}$, define $LF(x,\mathcal{U}) = \{e \in E : e < (x,0)\ and\ \exists m \geq 1\ with\ (e,(x,\frac{1}{m})) \prec \mathcal{U}\}$ and analogously define $LG(x,\mathcal{U}),\ LD(\mathcal{U})$ and

$$r_4(\mathcal{U}) = \{(e,(x,\frac{1}{m})) : x \in LD(\mathcal{U}),\ e \in LF(x,\mathcal{U}),\ (e,(x,\frac{1}{m})) \prec \mathcal{U}\}.$$

Then $r(\mathcal{U}) = \bigcup\{r_i(\mathcal{U}) : 1 \leq i \leq 4\}$ is a countable open cover of X that refines $\mathcal{U}$, and as in the proof of Proposition 3.1, the operator $\mathcal{U} \to r(\mathcal{U})$ is monotonic. □

The previous example is the first step in an inductive proof that for all finite $n \geq 1$, the lexicographic hypercube $[0,1]^n$ is monotonically Lindelöf. Part of the induction step from n to $n+1$ involves knowing that the open set $V(x) = \{(x,t_2,\cdots,t_{n+1}) : t_i \in [0,1]\} - \{(x,0,0,\cdots,0),(x,1,1,\cdots,1)\}$ of $[0,1]^{n+1}$ is monotonically Lindelöf. That follows from the general fact that if a space $Y = \bigcup\{Y_n : n < \omega\} = \bigcup\{int_Y(Y_n) : n < \omega\}$, where each Y_n is monotonically Lindelöf, then Y is also monotonically Lindelöf.

Example 2.6 *Let $(T,\leq_T)$ be an ω-branching Aronszajn tree with the property that nodes at limit levels are singletons. Order each non-limit-level node to make it a copy of $\mathbb{Z}$ and let X be the resulting branch space. Then X is monotonically Lindelöf.*

Proof: We know that the branch space X is Lindelöf [1]. For each $t \in T$, the set $[t] = \{b \in X : t \in b\}$ is a closed and open set. For any open cover $\mathcal{U}$ of X, we will say that an element $t \in T$ is $\mathcal{U}$-minimal if $[t]$ is a subset of some member of $\mathcal{U}$ and if for each $s <_T t$, the set $[s]$ is not a subset of any member of $\mathcal{U}$. Then the set $M(\mathcal{U}) = \{t \in T : t\ is\ \mathcal{U} - minimal\}$ is an antichain and $[t_1] \cap [t_2] = \emptyset$ whenever t_1 and t_2 are distinct and $\mathcal{U}$-minimal. Let $r(\mathcal{U}) = \{[t] : t \in M(\mathcal{U})\}$. Then $r(\mathcal{U})$ is a pairwise disjoint open cover of the Lindelöf space X, so $r(\mathcal{U})$ is countable. It is clear that if $\mathcal{U}$ refines $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. □

3 Separability and the Hereditary Monotone Lindelöf Property

Proposition 3.1 *Any separable GO-space is hereditarily monotonically Lindelöf.*

Proof: Let X be a separable GO-space. Because any subspace of X is again a separable GO-space, it is enough to show that X is monotonically Lindelöf. Fix a countable dense subset E of X. Let I be the set of all isolated points of X. Let $R = \{x \in X - I : [x,\rightarrow)\ is\ open\}$ and $L = \{x \in X - I : (\leftarrow,x]\ is\ open\}$.

For a set S and an open cover $\mathcal{U}$ of X, we will write $S \prec \mathcal{U}$ to mean that $S \subseteq U$ for some $U \in \mathcal{U}$. For any open cover $\mathcal{U}$ of X, we will define the refinement $r(\mathcal{U})$ in four steps. Let $r_1(\mathcal{U}) = \{\{x\} : x \in I\}$ and

$$r_2(\mathcal{U}) = \{(e_1,e_2) : e_i \in E\ and\ (e_1,e_2) \prec \mathcal{U}\}.$$

For any $x \in R$ define

$$RF(x,\mathcal{U}) = \{q \in E : x < q\ and\ [x,q) \prec \mathcal{U}\}$$

and

$$RG(x,\mathcal{U}) = \{q \in E : \exists y < x\ with\ (y,x) \neq \emptyset\ and\ (y,q) \prec \mathcal{U}\}.$$

Then $RG(x,\mathcal{U}) \subseteq RF(x,\mathcal{U})$. Let $RD(x,\mathcal{U}) = RF(x,\mathcal{U}) - RG(x,\mathcal{U})$ and define

$$r_3(\mathcal{U}) = \{[x,q) : x \in R \text{ and } RD(x,\mathcal{U}) \neq \emptyset \text{ and } q \in RF(x,\mathcal{U})\}.$$

Then $r_3(\mathcal{U})$ is a collection of open subsets of X.

We claim that $r_3(\mathcal{U})$ is countable. Because E is countable, it will suffice to show that the set $R(\mathcal{U}) = \{x \in R : RD(x,\mathcal{U}) \neq \emptyset\}$ is countable. For each $q \in E$ let $W(q) = \{x \in R : q \in RD(x,\mathcal{U})\}$. Then $R(\mathcal{U}) \subseteq \bigcup\{W(q) : q \in E\}$ so our claim will be verified if we show that $|W(q)| \leq 2$ for each $q \in E$. For contradiction, suppose there exist x_1, x_2, x_3 in some set $W(q)$. We may assume $x_1 < x_2 < x_3$. Then $q \in RF(x_1,\mathcal{U})$ implies that $[x_1, q) \prec \mathcal{U}$. But then $x_2 \in (x_1, x_3)$ and $(x_1, q) \prec \mathcal{U}$ showing that $q \in RG(x_3,\mathcal{U})$ so $q \notin RD(x_3,\mathcal{U})$. Thus $|W(q)| \leq 2$ for each $q \in E$.

Using L in place of R, for each $x \in L$ we define sets $LF(x,\mathcal{U})$, $LG(x,\mathcal{U})$, and $LD(x,\mathcal{U})$ and a countable collection $r_4(\mathcal{U}) = \{(q,x] : q < x,\ q \in E,\ \text{and } LD(x,\mathcal{U}) \neq \emptyset\}$ of open subsets of X. Define

$$r(\mathcal{U}) = r_1(\mathcal{U}) \cup r_2(\mathcal{U}) \cup r_3(\mathcal{U}) \cup r_4(\mathcal{U}).$$

Then $r(\mathcal{U})$ is countable.

We claim that $r(\mathcal{U})$ covers X. For suppose $x \in X$. If $x \in I$ then x is covered by $r_1(\mathcal{U})$. If $x \in X - (I \cup R \cup L)$, then choose any $U \in \mathcal{U}$ with $x \in U$. Then there are points $e_1 < x < e_2$ with $e_i \in E$ and $(e_1, e_2) \prec \mathcal{U}$, so that $(e_1, e_2) \in r_1(\mathcal{U})$ and hence x is covered by $r(\mathcal{U})$. It remains to consider points of $R \cup L$. Suppose $x \in R$, the other case being analogous, and suppose x is not covered by $r_1(\mathcal{U})$. Then $RG(x,\mathcal{U}) = \emptyset$ so that $RD(x,\mathcal{U}) \neq \emptyset$. Choose any $U \in \mathcal{U}$ with $x \in U$. Because $x \in R \subseteq X - I$ we know that some $q \in E$ has $x < q$ and $[x,q) \subseteq U$. Then $[x,q) \in r_3(\mathcal{U})$ so that $r(\mathcal{U})$ covers x.

Finally, suppose $\mathcal{U}$ and $\mathcal{V}$ are open covers of X and that $\mathcal{U}$ refines $\mathcal{V}$. Clearly $r_i(\mathcal{U}) \subseteq r_i(\mathcal{V})$ for $i = 1, 2$, so suppose $[x,q) \in r_3(\mathcal{U})$. Then $x \in R,\ RD(x,\mathcal{U}) \neq \emptyset$ and $[x,q) \prec \mathcal{U}$. If $RD(x,\mathcal{V}) \neq \emptyset$ then $[x,q) \in r_3(\mathcal{V})$, as required. If $RD(x,\mathcal{V}) = \emptyset$ then $RF(x,\mathcal{V}) = RG(x,\mathcal{V})$. Note that $q \in RF(x,\mathcal{U}) \subseteq RF(x,\mathcal{V})$ so we know that $q \in RG(x,\mathcal{V})$. Hence there is some $y < x$ with $(y,x) \neq \emptyset$ and $(y,q) \prec \mathcal{V}$. Choose any $e_1 \in E \cap (y,x)$. Then $(e_1,q) \prec \mathcal{V}$ so that $[x,q) \subseteq (e_1,q) \in r_1(\mathcal{V}) \subseteq r(\mathcal{V})$. Similarly, any set $(q,x] \in r_4(\mathcal{U})$ is either a member or, or a subset of some member of, $r(\mathcal{V})$. Therefore $r(\mathcal{U})$ refines $r(\mathcal{V})$, as required. $\square$

The proof above actually shows more. If we modify the definition of monotone Lindelöf to refer to the existence of refinements of cardinality κ and call the resulting property *monotone κ-Lindelöf*, the proof of Proposition 3.1 shows that if X is a GO-space with $d(X) = \kappa$, then X is monotonically κ-Lindelöf.

An often-used technique for constructing examples is to begin with a LOTS $(X, <, \mathcal{S})$, select three disjoint subsets R, L and I of X, and then isolate all points in I while changing the neighborhood systems of points of $x \in R$ and $y \in L$ to make sets of the form $[x,b)$ and $(a,y]$ open in a new topology $\mathcal{T}$ on X. This process is called *constructing a GO-space on* $(X, <, \mathcal{S})$. The construction of the Sorgenfrey line and the Michael line are perhaps the best known examples of this construction. The above proof can be modified to show:

Proposition 3.2 *Suppose that the LOTS $(X, <, \mathcal{S})$ is separable and that $\mathcal{T}$ is a GO-topology constructed on $(X, <, \mathcal{S})$. Then the following are equivalent:*

a) $(X, \mathcal{T})$ *is Lindelöf;*

b) for any $\mathcal{T}$*-open set* U *containing the set* X^d *of all non-isolated points of* $(X, \mathcal{T})$*, the set* $X - U$ *is countable;*

c) $(X, \mathcal{T})$ *is monotonically Lindelöf.* $\Box$

Does the converse of Proposition 3.1 hold? Is it true that the monotone Lindelöf property characterizes separability among GO-spaces? It is no surprise to find that this is a Souslin problem issue so that consistent answers are the best we can hope for.

Proposition 3.3 *Let* M *be any model of* ZFC*. If* M *contains a Souslin line, then* M *has a GO-space that is hereditarily monotonically Lindelöf but not separable, and if* M *does not contain a Souslin line, then separability is equivalent to the hereditary monotone Lindelöf property for any GO-space in* M*.*

Proof: First, suppose M contains a Souslin line. Then M contains an ω-branching Souslin tree T. Order each node of T making it a copy of $\mathbb{Z}$ and let X be the resulting branch space. For each $t \in T$, the set $[t] = \{b \in X : t \in b\}$ is closed and open in X. Let Y be any subspace of X and let $\mathcal{U}$ be any relatively open cover of Y. We will say that a $t \in T$ is $\mathcal{U}$-minimal if $[t] \cap Y$ is contained in some member of $\mathcal{U}$ and if $s <_T t$, then $[s] \cap Y$ is not contained in any member of $\mathcal{U}$. Then the set $M(\mathcal{U})$ consisting of all $\mathcal{U}$-minimal points of T is an anti-chain of T, so that $M(\mathcal{U})$ is countable. Defining $r(\mathcal{U}) = \{[t] \cap Y : t \in M(\mathcal{U})\}$, we obtain a countable relatively open cover of Y, and clearly r has the required monotone property.

Second, suppose M contains no Souslin line. Proposition 3.1 shows that if X is a separable GO-space, then X is hereditarily monotonically Lindelöf. For the converse, suppose X is a GO-space that is hereditarily monotonically Lindelöf. Then X has countable cellularity so that because no Souslin lines exist in M, X must be separable. $\Box$

4 Open Questions

There are several examples (or theorems) still needed to round out the elementary theory of the monotone Lindelöf property in GO-spaces.

1) Is there a compact first countable GO-space that is not monotonically Lindelöf?

2) If there is a Souslin line, is there a compact Souslin line that is (hereditarily) monotonically Lindelöf?

3) If there is a Souslin line, is there a Souslin line that is not monotonically Lindelöf?

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