Topology Inside ω_1

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Abstract: In this expository paper, we show how the Pressing Down Lemma and Ulam matrices can be used to study the topology of subsets of ω_1. We prove, for example, that if S and T are stationary subsets of ω_1 with $S\Delta T = (S - T) \cup (T - S)$ stationary, then S and T cannot be homeomorphic. Because Ulam matrices provide ω_1-many pairwise disjoint stationary subsets of any given stationary set, it follows that there are 2^{ω_1}-many stationary subsets of any stationary subset of ω_1 with the property that no two of them are homeomorphic to each other. We also show that if S and T are stationary sets, then the product space $S \times T$ is normal if and only if $S \cap T$ is stationary. In addition, we prove that for any $X \subseteq \omega_1$, $X \times X$ is normal, and that if $X \times X$ is hereditarily normal, then $X \times X$ is metrizable.

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1 Introduction

In this expository paper, we explore topology inside of ω_1 using set theoretic tools of stationary sets. We make no claim of novelty for these results: any logician or topologist will know at least half of them.

The space ω_1 is the well-ordered set of all countable ordinals with the open-interval topology of that well-ordering. It is a standard example in general topology courses, a space that is (for example) countably compact but not compact, first countable but not second countable, normal but not paracompact, and locally metrizable but not metrizable. In addition, the set ω_1 and some of its larger relatives are standard tools in modern set theory.

In Section 2 we begin with a whirlwind tour of club-sets, stationary sets, the Pressing Down Lemma, and Ulam matrices. Then we show how such set-theoretic tools can be used to study some more advanced topics in the topology of subspaces of ω_1. For example, in Section 3 we prove:

a) When ω_1 carries the open-interval topology of its linear order, any subspace of ω_1 is either stationary or metrizable.

b) If two stationary sets S and T are homeomorphic as topological subspaces of ω_1, then $S \cap T$ is stationary and $S \Delta T$ is not stationary so that there are 2^{ω_1} many different topological types of stationary subsets of ω_1.

Section 4 of our paper uses the Pressing Down Lemma to study normality of the product of two stationary sets and we prove:

c) For two stationary subsets S and T of ω_1, the product space $S \times T$ is normal if and only if $S \cap T$ is stationary, and for any subset X of ω_1, $X \times X$ is normal.

We conclude with an annotated list of some other results about the topology of stationary sets.

In this paper, we give proofs using only the tools outlined in Section 2 even though shorter proofs exist using more esoteric ideas from general topology. We thank several colleagues, and especially N. Kemoto, for comments that substantially improved our exposition. Throughout this paper, we assume the Axiom of Choice(AC), except where specifically noted.

2 Basic tools: the set ω_1, club-sets, and stationary sets

The results in this section appear in any modern textbook on mathematical logic. Using the Axiom of Choice(AC), it is easy to prove that there is a unique (up to order isomorphism) uncountable well-ordered set called ω_1 with the special property that for each member β of ω_1 the set of predecessors of β is countable[1]. By working harder one can get ω_1 without AC [9]. In an abuse of notation, we sometimes think of ω_1 as the right endpoint of the set, and write things like $[\alpha, \omega_1) := \{\beta \in \omega_1 : \alpha \leq \beta\}$.

Any nonempty countable subset of ω_1 has a least upper bound in ω_1. In the open interval topology of the given well-ordering $\leq$ of ω_1 a subset C of ω_1 is closed if and only if $\sup(D) \in C$ for each countable subset $D \subseteq C$. The set of all limit ordinals in ω_1, i.e. the set of all $\lambda \in \omega_1$ that have no immediate predecessor, is an example of a closed set that is uncountable (equivalently, unbounded). Such sets are called "club-sets" (or sometimes "cub-sets"). The next three results give ways to produce more complicated club-sets.

Proposition 2.1 *If $C_0, C_1, C_2, \cdots$ is a sequence of club-sets, then $\bigcap\{C_n : n < \omega\}$ is also a club-set.*

Proof: It is clear that $\bigcap\{C_n : n < \omega\}$ is a closed set. To show that $\bigcap\{C_n : n < \omega\}$ is unbounded, line up the sets C_n as follows:

$$C_0, C_1;\ C_0, C_1, C_2;\ C_0, C_1, C_2, C_3;\ C_0, \cdots$$

and fix any $\gamma \in \omega_1$. Choose any $\alpha_0 \in C_0$ with $\alpha_0 > \gamma$. Given α_n, choose α_{n+1} from the $(n+1)^{st}$ set in the list with $\alpha_{n+1} > \alpha_n$. The fact that each C_n is unbounded makes this possible. Let $\beta = \sup(\{\alpha_n : n < \omega\})$. Then $\beta \in \bigcap\{C_n : n < \omega\}$ and $\beta > \gamma$ as required. $\square$

Proposition 2.2 *(Diagonal Intersection) Suppose D_α is a club-set for each $\alpha < \omega_1$. Then the set $E = \{\beta < \omega_1 : \beta \in \bigcap\{D_\alpha : \alpha < \beta\}\}$ is also a club-set.*

[1] Today this set is usually called ω_1, but older books sometimes call it Ω (see [10])

Proof: It is easy to check that E is closed, but upon first inspection it is not clear that the set E is non-empty, let alone unbounded. So suppose $\gamma_0 < \omega_1$ is fixed. By Proposition 2.1, the set $\bigcap\{D_\alpha : \alpha < \gamma_0\}$ is a club-set, so let γ_1 be the first element of $\bigcap\{D_\alpha : \alpha < \gamma_0\}$ with $\gamma_1 > \gamma_0$. Given γ_n, the set $\bigcap\{D_\alpha : \alpha < \gamma_n\}$ is a club-set, so we let γ_{n+1} be the first member of that set that is greater than γ_n. Let $\delta = \sup\{\gamma_n : n < \omega\}$. Then $\delta \in E$ and $\delta > \gamma_0$, so that E is an unbounded subset of ω_1 as required. $\square$

Proposition 2.3 *Suppose $S \subseteq \omega_1$ and $f : S \to \omega_1$ has $\alpha < f(\alpha)$ for all $\alpha \in S$. Then there is a club-set C such that if $\gamma \in C$ and $\alpha \in S$ has $\alpha < \gamma$, then $f(\alpha) < \gamma$. In particular, if $\alpha_1 < \alpha_2$ are in $S \cap C$, then $f(\alpha_1) < f(\alpha_2)$.*

Proof: Let $C := \{\gamma \in \omega_1 : \text{if } \alpha \in S \text{ and } \alpha < \gamma \text{ then } f(\alpha) < \gamma\}$. To show that C is nonempty and unbounded, fix any $\alpha_0 < \omega_1$. Given α_n, note that the set $F_n := \{f(\beta) : \beta \in S \text{ and } \beta \leq \alpha_n\}$ is countable, so there is some $\alpha_{n+1} > \alpha_n$ with $F_n \subseteq [0, \alpha_{n+1})$. Let $\gamma = \sup(\{\alpha_n : n < \omega\})$. Note that if $\alpha \in S$ and $\alpha < \gamma \in C$, then for some $n, \alpha < \alpha_n$ so that $f(\alpha) < \alpha_{n+1} < \gamma$, showing that $\gamma \in C$. Because $\gamma > \alpha_0$, the set C is nonempty and unbounded. A similar argument shows that if $\gamma_0 < \gamma_1 < \cdots$ are points of C, then $\sup\{\gamma_n : n < \omega\} \in C$, so C is closed. Now suppose $\alpha_1 < \alpha_2$ are in $C \cap S$. Then $f(\alpha_1) < \alpha_2 < f(\alpha_2)$ as required. $\square$

Almost any set of ordinals that one can describe will either contain a club-set, or its complement will contain a club-set. Such sets form an important class of subsets of ω_1. Recall that the family of Borel sets in a space X, called $Borel(X)$, is the smallest σ-algebra of subsets of X that contains every closed set. In the case of ω_1, we have:

Proposition 2.4 *The collection $Borel(\omega_1)$ is the collection of all sets $S \subseteq \omega_1$ where either S or $\omega_1 - S$ contains a club-set.*

Proof: Write $\mathcal{C} = \{S \subseteq \omega_1 : \text{either } S \text{ or } \omega_1 - S \text{ contains a club-set}\}$. Using Proposition 2.1 it is easy to check that $\mathcal{C}$ is a σ-algebra and that $\mathcal{C}$ contains every closed subset of ω_1, so $Borel(\omega_1) \subseteq \mathcal{C}$. Next consider any subset S of ω_1 where $C \subseteq S$ for some club-set C. Then $\omega_1 - C = \bigcup\{U_i : i \in I\}$ where $\{U_i : i \in I\}$ is a pairwise disjoint collection of non-empty countable, convex[2] open sets. Index each $U_i = \{\alpha(i,n) : n < \omega\}$, with repetitions allowed in case some U_i is finite. Let $D_n = \{\alpha(i,n) : i \in I\}$ and let $E_n = cl(D_n)$. Both E_n and $\omega_1 - C$ are in $Borel(\omega_1)$ so that $D_n = E_n \cap (\omega_1 - C) \in Borel(\omega_1)$. Therefore $S = C \cup (\bigcup\{D_n : n < \omega\}) \in Borel(\omega_1)$ as claimed. Finally, if $\omega_1 - S$ contains a club-set, the above argument shows that $\omega_1 - S \in Borel(\omega_1)$ so that $S \in Borel(\omega_1)$. $\square$

One might wonder whether there are any subsets of ω_1 except for the Borel sets. Under (AC), the answer is "Yes" as we will show below. But first, we need some terminology. A subset $S \subseteq \omega_1$ is said to be a *stationary set* if $S \cap C \neq \emptyset$ for each club-set $C \subseteq \omega_1$. In the light of Proposition 2.4, the non-Borel subsets of ω_1 (if there are any) are precisely the sets $T \subseteq \omega_1$ with the property that both T and $\omega_1 - T$ are stationary, and such sets are said to be *bi-stationary.* Using AC, Mary Ellen Rudin gave a particularly elegant proof that ω_1 must have bi-stationary subsets. What follows is a slight re-writing of Rudin's proof.

[2]The set C is *convex* if $\alpha < \beta < \gamma$ and $\alpha, \gamma \in C$ imply that $\beta \in C$. A maximal convex subset of a set D is called a *convex component* of D. Any subset of ω_1 is the pairwise disjoint union of its maximal convex subsets, which are called *convex components* of the set.

Proposition 2.5 *There is a bi-stationary subset $S \subseteq \omega_1$, i.e., a set S with the property that neither S nor $\omega_1 - S$ contains a club-set.*

Proof: For contradiction, suppose that for every subset $S \subseteq \omega_1$, either S or $\omega_1 - S$ contains a club-set. In the set $\mathbb{R}$ of real numbers, let $\mathcal{D}_0 = \{[n, n+1) : n \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of all integers. Let $\mathcal{D}_1 = \{[n, n+\frac{1}{2}) : n \in \mathbb{Z}\} \cup \{[n+\frac{1}{2}, n+1) : n \in \mathbb{Z}\}$. Continue this bisection process, creating for each $n < \omega$ a countable pairwise disjoint collection $\mathcal{D}_n$ of half-open intervals of $\mathbb{R}$ that has $\bigcup \mathcal{D}_n = \mathbb{R}$, and where every member of $\mathcal{D}_n$ has length $\frac{1}{2^n}$.

Because $|\omega_1| \leq 2^\omega$, there is a function $f : \omega_1 \to \mathbb{R}$ that is an injection (but possibly not a surjection).

Now fix $n < \omega$. We claim that for some $D \in \mathcal{D}_n$, the set $f^{-1}[D]$ contains a club-set. If that is not true, then for each $D \in \mathcal{D}_n$, the set $\omega_1 - f^{-1}[D]$ must contain a club-set C_D. But then $\bigcap\{\omega_1 - f^{-1}[D] : D \in \mathcal{D}_n\}$ contains the club-set $\bigcap\{C_D : D \in \mathcal{D}_n\}$ by Proposition 2.1. Therefore $\bigcap\{\omega_1 - f^{-1}[D] : D \in \mathcal{D}_n\} \neq \emptyset$. Let α be the first member of $\bigcap\{\omega_1 - f^{-1}[D] : D \in \mathcal{D}_n\}$. But then $f(\alpha) \in \mathbb{R}$ and $f(\alpha) \notin D$ for every $D \in \mathcal{D}_n$, and that is impossible because $\bigcup \mathcal{D}_n = \mathbb{R}$.

In the light of the previous paragraph, for each n, we may choose a member $D_n \in \mathcal{D}_n$ that has $f^{-1}[D_n] \supseteq C_n$ where C_n is a club-set (we note that there is exactly one such D_n). Then Proposition 2.1 shows that $\bigcap\{C_n : n < \omega\}$ is a club-set, so we can let α and β be the first two points of that set. But then $f(\alpha), f(\beta) \in D_n$ for each n so that for each n the distance between $f(\alpha)$ and $f(\beta)$ is less than the length of D_n which is $\frac{1}{2^n}$. Therefore, $f(\alpha) = f(\beta)$ and that is impossible because f is injective. $\square$

Stationary sets share many properties of ω_1 as can be seen from the next lemma whose proof uses Propositions 2.1 and 2.2 and whose results will be needed in Section 4.

Lemma 2.6 *Let $S \subseteq \omega_1$ be a stationary set.*

a) If $S = \bigcup\{S_n : n < \omega\}$, then one of the sets S_n is stationary;

b) If C_n is a relatively closed uncountable subset of S (to be called a relative club-set in S) for each $n < \omega$, then $\bigcap\{C_n : n < \omega\}$ is also a relative club-set in S;

c) If D_α is a relative club-set in S for each $\alpha \in S$, then $\{\alpha \in S : \text{ for all } \beta < \alpha, \alpha \in D_\beta\}$ is a relative club-set in S.

$\square$

Rudin's proof in Proposition 2.5 can be modified to show that each stationary set can be split into two disjoint stationary sets (use an injective function from a stationary set into $\mathbb{R}$), but in the next section we need much more, so we turn to an elegant tool called an *Ulam matrix.* Ulam introduced this matrix to solve a measure-theoretic problem (discussed in Remark 2.11 below) that had nothing to do with stationary sets. An *Ulam matrix* is a special array of sets $U(\alpha, n)$ with ω_1 rows and ω columns. Ulam was answering a question about measures defined in ω_1 and gave his construction [17] for the set $S = \omega_1$, but because we need a corollary about splitting stationary sets (see Corollary 3.4 below), we present Ulam's proof in a more general format. See [14] for a more complete discussion.

Lemma 2.7 *(Ulam) Let S be a stationary subset of ω_1. For each $\alpha \in S$ the set $S \cap [0, \alpha]$ is countable so there is a surjection $f_\alpha : \omega \to S \cap [0, \alpha]$. Define $U(\alpha, n) = \{\beta \geq \alpha : \beta \in S \text{ and } f_\beta(n) = \alpha\}$. Then*

(i) $S \cap [\alpha, \omega_1) = \bigcup\{U(\alpha, n) : n < \omega\}$, and

(ii) if $\alpha_1 \neq \alpha_2$ then for each $n < \omega, U(\alpha_1, n) \cap U(\alpha_2, n) = \emptyset$.

Proof: To verify (i), suppose $\gamma \in S \cap [\alpha, \omega_1)$. Then $f_\gamma : \omega \to S \cap [0, \gamma]$ is a surjection and $\alpha \in S \cap [0, \gamma]$ so there is some $n < \omega$ with $f_\gamma(n) = \alpha$. But then $\gamma \in U(\alpha, n)$. It is clear that every $U(\alpha, n) \subseteq S \cap [\alpha, \omega_1)$ so that (i) is established.

To prove (ii), suppose $\gamma \in U(\alpha_1, n) \cap U(\alpha_2, n)$ with $\alpha_1 \neq \alpha_2$. Then $f_\gamma(n) = \alpha_1$ and $f_\gamma(n) = \alpha_2$, and that is impossible because f_γ is a function. $\square$

We will need the next corollary in Section 3 when we find the number of topologically different stationary sets.

Corollary 2.8 *Given any stationary subset S of ω_1, there is an uncountable pairwise disjoint collection Φ of subsets of S where each member of Φ is stationary in ω_1.*

Proof: In the Ulam matrix above, for each $\alpha < \omega_1$ we have $S \cap [\alpha, \omega_1) = \bigcup\{U(\alpha, n) : n < \omega\}$ so that Lemma 2.6 gives an integer n_α for which $U(\alpha, n_\alpha)$ is stationary. There are only countably many integers, so there is an integer k such that the set $B = \{\beta < \omega_1 : n_\beta = k\}$ is uncountable. Now apply (ii) in Lemma 2.7 to conclude that if $\beta_1 \neq \beta_2$ in B, then $U(\beta_1, k) \cap U(\beta_2, k) = \emptyset$. Then $\Phi = \{U(\beta, k) : \beta \in B\}$ is the required pairwise disjoint uncountable collection. $\square$

The following result will be needed in Section 4 of our paper.

Corollary 2.9 *Suppose S is a stationary subset of ω_1 and suppose $f : S \to \omega_1$ has $\alpha < f(\alpha)$ for all $\alpha \in S$. Then there is a club-set C with the property that if α_1 and α_2 both belong to the stationary set $S \cap C$ and have $\alpha_1 < \alpha_2$, then $f(\alpha_1) < f(\alpha_2)$.*

Proof: Find the set C as in Proposition 2.3. Then $S \cap C$ is also a stationary set and if $\alpha_1 < \alpha_2$ both belong to $S \cap C$ then $f(\alpha_1) < \alpha_2 < f(\alpha_2)$. $\square$

The central tool for the study and application of stationary sets is called the *Pressing Down Lemma*, and is also known as Fodor's lemma.

Lemma 2.10 *Suppose S is a stationary subset of ω_1 and $f : S \to \omega_1$ has $f(\alpha) < \alpha$ for each $\alpha \in S$. Then there is a stationary subset $T \subseteq S$ and an ordinal $\beta < \omega_1$ with the property that $f(\alpha) = \beta$ for each $\alpha \in T$. Consequently, the fiber $f^{-1}[\{\beta\}]$ of f is stationary.* $\square$

Proof: Suppose not. Then for each $\beta < \omega_1$, the set $f^{-1}[\{\beta\}]$ is not stationary, so there is a club-set D_β that is disjoint from $f^{-1}[\{\beta\}]$. From Proposition 2.2 we know that the set $E = \{\gamma < \omega_1 : \gamma \in \bigcap\{D_\beta : \beta < \gamma\}\}$ is a club-set, so there is some $\alpha \in S \cap E$. Write $\delta = f(\alpha)$. Then $\alpha \in f^{-1}[\{\delta\}]$ and because $\delta = f(\alpha) < \alpha$ we know that $\alpha \in \bigcap\{D_\gamma : \gamma < \alpha\} \subseteq D_\delta$. But that is impossible because $D_\delta \cap f^{-1}[\{f(\alpha)\}] = \emptyset$. $\square$.

Remark 2.11 *Measure theory in ω_1.* In Proposition 2.4 we showed that a subset $S \subseteq \omega_1$ is a Borel set in ω_1 if and only if either S or $\omega_1 - S$ contains a club-set. This allows us to define a measure m whose domain is $Borel(\omega_1)$ by the rule that for $S \in Borel(\omega_1)$, $m(S) = 1$ if S contains a club-set, and $m(S) = 0$ otherwise. This m is a countably additive bounded measure and is non-atomic, i.e., $m(\{\alpha\}) = 0$ for each $\alpha \in \omega_1$. The question is: could we extend the domain of m to be the entire power set of ω_1? Ulam presented a very elegant solution in [17] by using the matrix in Lemma 2.7 to show that there is no non-trivial countably additive, finite-valued, non-atomic, bounded measure whose domain is the power set of ω_1 . See [14] for a good discussion. It is an amusing exercise to figure out why the following construction does not contradict Ulam's result. Let $\mathcal{U}$ be a free ultrafilter[3] on ω_1. Define $m(S) = 1$ if $S \in \mathcal{U}$ and $m(S) = 0$ otherwise. Explain why this m does not violate Ulam's Theorem about the non-existence of measures whose domain is the power set of ω_1.

Remark 2.12 Most mathematicians use the Axiom of Choice, but there is another axiom system (incompatible with AC called "ZF + AD" where AD stands for the Axiom of Determinacy. In ZF + AD one can prove that every subset of ω_1 either contains or is disjoint from a club-set, and consequently the power set of ω_1 is the collection $Borel(\omega_1)$ and therefore the $\{0, 1\}$-valued measure m at the beginning of this section is defined for all subsets of ω_1.

3 Basic topology of stationary sets

Any subset $S \subseteq \omega_1$ inherits a subspace topology from ω_1 whose open sets are of the form $S \cap V$ where V is an open subset of ω_1, and any such set is called "relatively open in S." (Warning: This is usually not the same as the open-interval-topology defined on S by the restriction of the well-ordering $\leq$ to S.) We begin with an internal characterization of stationary subsets of ω_1.

Proposition 3.1 *For any subset $S \subseteq \omega_1$, the following are equivalent:*

a) S is not stationary;

b) there is a collection $\mathcal{U}$ of pairwise disjoint relatively open subsets of S that covers S, where each $U \in \mathcal{U}$ is countable;

c) in its subspace topology, S is metrizable.

Consequently, any subspace of ω_1 is either metrizable or stationary.

Proof: Recall that a subset C of any linearly ordered set $(X, <)$ is *convex* provided $a, b \in C$ and $a < x < b$ gives $x \in C$. The convex components of a subset $D \subseteq X$ are the maximal convex subsets of D, and D is the pairwise disjoint union of its convex components.

To show that a) implies b), suppose S is not stationary. Then there is a club-set C with $S \subseteq \omega_1 - C$. Express $\omega_1 - C$ as the union of its convex components, say $\omega_1 - C = \bigcup\{U_i : i \in I\}$.

[3]A free ultrafilter on ω_1 is a nonempty collection $\mathcal{U}$ of infinite subsets of ω_1 having four properties: (i) if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$; (ii) if $A \in \mathcal{U}$ and $A \subseteq B \subseteq \omega_1$, then $B \in \mathcal{U}$; (iii) for every set $S \subseteq \omega_1$, either $S \in \mathcal{U}$ or $\omega_1 - S \in \mathcal{U}$; and (iv) $\bigcap \mathcal{U} = \emptyset$.

Then each U_i is an open subset of ω_1, $U_i \cap U_j = \emptyset$ if $i \neq j$, each U_i is countable, and we have $S = \bigcup\{S \cap U_i : i \in I\}$ as claimed.

To show that b) implies c), write $S = \bigcup\{V_i : i \in I\}$ where each V_i is a countable relatively open subset of S, and the sets V_i are pairwise disjoint. Each V_i, being a regular, countable and first-countable space, is metrizable by the Urysohn metrization theorem [13]. Then S, being a union of pairwise disjoint open metrizable spaces, is a metric space, as claimed.

To show that c) implies a), suppose for contradiction that c) holds and S is stationary. Let T be the set of relatively non-isolated points of S, i.e., T is the set of limit points of S that belong to S. Then T is also stationary and is a relatively closed subset of S. Because S is a metric space, T is a G_δ-set in the space S so there are relatively open subsets $U(n)$ of S with $T = \bigcap\{U(n) : n < \omega\}$. Fix n. For each $\lambda \in T$, there is an ordinal $f_n(\lambda) < \lambda$ with $S \cap (f_n(\lambda), \lambda] \subseteq U(n)$. The Pressing Down Lemma 2.10 gives an ordinal β_n and a stationary set T_n with the property that $f_n(\lambda) = \beta_n$ for each $\lambda \in T_n$. Then $(\beta_n, \omega_1) \cap S \subseteq U(n)$. Having found the ordinals β_n we let $\gamma = \sup\{\beta_n : n < \omega\}$ and we see that $(\gamma, \omega_1) \cap S \subseteq \bigcap\{S \cap U_n : n < \omega\} = T$. But that is impossible because the first element of S greater than γ belongs to $(\gamma, \omega_1) \cap S$ but not to T. $\square$

Corollary 3.2 *Suppose S is a stationary subset of ω_1 and $g : S \to \omega_1$ is continuous. Write $T = g[S]$. Then the following are equivalent:*

1) T is stationary;

2) T is uncountable;

3) each fiber $g^{-1}[\beta]$ of g is countable.

Proof: Clearly 1) implies 2). To prove that 2) implies 3), suppose the set T is uncountable and yet for some β, the set $A = g^{-1}[\beta]$ is uncountable. Then A, being an uncountable relatively closed subset of the stationary set S, is also stationary. Because the space ω_1 is first-countable, the set $\{\beta\}$ is a G_δ-subset of T, so the set A is a G_δ-subset of S, say $A = \bigcap\{U_n : n \geq 1\}$ where each U_n is a relatively open subset of S. For each n and each $\alpha \in A$ there is an ordinal $f_n(\alpha) < \alpha$ with $(f_n(\alpha), \alpha] \cap S \subseteq U_n$. The Pressing Down Lemma gives some δ_n with $f_n(\alpha) = \delta_n$ for all α in a stationary subset T_n of T, so that $(\delta_n, \alpha] \cap S \subseteq U_n$ for all $\alpha \in T_n$, and therefore $(\delta_n, \omega_1) \cap S \subseteq U_n$. Let $\delta = \sup\{\delta_n : n \geq 1\}$. Then $(\delta, \omega_1) \cap S \subseteq \bigcap\{U_n \cap S : n \geq 1\} = A$. But then $T = g[S] = \{\beta\} \cup \{g(\alpha) : \alpha \in S \cap [0, \delta]\}$ showing that T is a countable set, contradicting 2).

To complete the proof we show that 3) implies 1), so suppose 3) holds and yet T is not stationary. Then Proposition 3.1 shows that there is a pairwise disjoint collection $\{V_j : j \in J\}$ of countable, relatively open subsets of T that covers T. Then $\{g^{-1}[V_j] : j \in J\}$ is a pairwise disjoint relatively open cover of S, and we see that each $g^{-1}[V_j] = \bigcup\{g^{-1}[\{\beta\}] : \beta \in V_j\}$ is a countable set because each fiber of g is countable. In the light of Proposition 3.1, that is impossible. $\square$

Recall that two spaces X and Y are topologically the same if there is a homeomorphism h from S onto T, i.e., a bijection so that both h and h^{-1} are continuous. Could it happen that the disjoint stationary sets found in Theorem 2.5 are homeomorphic spaces? More generally, how many different topological types of stationary subsets of ω_1 exist? A partial answer was given in [4] as follows:

Proposition 3.3 *Suppose S and T are stationary subsets of ω_1 that are homeomorphic. Then $S \cap T$ is stationary and the symmetric difference $S \Delta T = (S - T) \cup (T - S)$ is not stationary.*

Proof: Suppose $h : S \to T$ is a homeomorphism. Let $S_1 = \{\alpha \in S : h(\alpha) < \alpha\}$, $S_2 = \{\alpha \in S : h(\alpha) = \alpha\}$ and $S_3 = \{\alpha \in S : h(\alpha) > \alpha\}$. Because $S = S_1 \cup S_2 \cup S_3$ is stationary, one of the sets S_i must be stationary by Lemma 2.6. The set S_1 cannot be stationary because h is injective and that would violate the Pressing Down Lemma (Lemma 2.10). Next consider S_3. If S_3 is stationary, then by Corollary 3.2 the set $T_3 = h[S_3]$ is also stationary because each fiber of h is a singleton. But T_3 cannot be stationary because $h^{-1} : T_3 \to S_3$ has $h^{-1}(\beta) < \beta$ for each $\beta \in T_3$, contrary to the Pressing Down Lemma. We conclude that S_3 cannot be stationary. Therefore S_2 is stationary, and because $S_2 \subseteq S \cap T$, the set $S \cap T$ is also stationary, as claimed.

Now consider the set $S - T$. Because $S_2 \subseteq S \cap T \subseteq T$, we see that $\omega_1 - T \subseteq \omega_1 - S_2$ so that $S - T \subseteq S - S_2 = S_1 \cup S_3$. Because S_1 and S_3 are non-stationary, so is $S - T$. Next consider the set $T - S$. Apply the first part of the proof to the homeomorphism $h^{-1} : T \to S$ to conclude that $T - S$ is also non-stationary. Therefore $S \Delta T$ is non-stationary. $\square$

Corollary 3.4 *Given any stationary set S in ω_1, there is a collection Ψ of subsets of S where*

(i) each member of Ψ is a stationary subset of ω_1

(ii) $|\Psi| = 2^{\omega_1}$

(iii) if $T_1 \neq T_2$ are members of Ψ, then T_1 and T_2 are not homeomorphic to each other.

Proof: Let Φ be the uncountable pairwise disjoint collection of stationary subsets of S found in Corollary 2.8. For each nonempty $\mathcal{C} \subseteq \Phi$ let $S(\mathcal{C}) = \bigcup \mathcal{C}$. Each $S(\mathcal{C})$ is stationary, and if $\mathcal{C}, \mathcal{D}$ are different non-void subcollections of Φ, we may choose $T \in \mathcal{C} - \mathcal{D}$ or $T \in \mathcal{D} - \mathcal{C}$. Then $T \subseteq S(\mathcal{C}) \Delta S(\mathcal{D})$ showing that $S(\mathcal{C}) \Delta S(\mathcal{D})$ is not non-stationary, so the subspaces $S(\mathcal{C})$ and $S(\mathcal{D})$ cannot be homeomorphic by Proposition 3.3. Then $\Psi := \{S(\mathcal{C}) : \emptyset \neq \mathcal{C} \subseteq \Phi\}$ is the required collection of size 2^{ω_1}. $\square$

Proposition 3.3 gives a necessary condition for two stationary subsets of ω_1 to be homeomorphic, but that condition is not sufficient. Consider the stationary sets $S = \omega_1$ and $T = S - \{\omega\}$. Clearly S and T are not homeomorphic (because T contains a sequence without a limit point while S does not) and yet $S \cap T$ is stationary and $S \Delta T$ is not stationary. However, the hypothesis $S \Delta T$ is non-stationary does have an interesting topological characterization. Recall that two spaces X and Y are *Borel isomorphic* if there is a bijection $g : X \to Y$ such that if C is a Borel set in X, then $g[C]$ is a Borel set in Y, and if D is a Borel set in Y, then $g^{-1}[D]$ is a Borel set in X. The following result is proved in [4]:

Proposition 3.5 *Let S and T be stationary subsets of ω_1. Then $S \Delta T$ is non-stationary if and only if S and T are Borel isomorphic.*

Readers who want to test their pressing down skills should prove the following, which shows that every continuous, real-valued function on a stationary set S must be "constant on a tail of S."

Proposition 3.6 *If $S \subseteq \omega_1$ is stationary and if $f : S \to \mathbb{R}$ is a continuous function, then there is some $\alpha \in S$ such that for each $\beta \in S \cap [\alpha, \omega_1)$, $f(\beta) = f(\alpha)$.* $\square$

4 The product of two stationary sets

Every topology student knows that the product of two Hausdorff spaces is a Hausdorff space, and the product of two regular spaces is again regular. But the product of two normal spaces is something entirely different, as can be seen by considering the product of two copies of the Sorgenfrey line. The goal of this section is to show how the stationary set theory outlined above can be used to study the product space $S \times T$ where S and T are stationary subsets of ω_1. We will see that normality of $S \times T$ depends entirely upon whether or not $S \cap T$ is stationary. As a consequence we will see that for every $X \subseteq \omega_1$, the space $X \times X$ is normal.

Throughout this section we use the following special notation. If $\alpha < \beta$ are points of ω_1 and $S \subseteq \omega_1$, then $[\alpha, \beta]_S = [\alpha, \beta] \cap S$ and other interval notations like $(\alpha, \beta]_S$ are similarly defined. It does not matter whether $\alpha, \beta \in S$. Similar comments apply to the notation $[\alpha, \beta]_T = [\alpha, \beta] \cap T$ and its variations. In addition we will use the term *clopen* for a set that is both closed and open. For example, any interval $(\alpha, \beta]$ is a clopen subset of ω_1.

Theorem 4.1 *For stationary subsets S and T of ω_1, the product space $S \times T$ is normal if and only if $S \cap T$ is stationary.*

Proof: The proof has two major parts which we present in Propositions 4.2 and 4.4. To avoid notational confusion between ordered pairs and open intervals, we will write $\langle \alpha, \beta \rangle$ for a point of the product $S \times T$ and reserve the notation (α, β) for an interval in ω_1.

Proposition 4.2 *If S and T are stationary subsets of ω_1 and if $S \cap T$ is not stationary, then $S \times T$ is not normal.*

Proof: The proof below was suggested by N. Kemoto in a private communication.

Because $S \cap T$ is not stationary, there is a club-set C_0 with $(S \cap T) \cap C_0 = \emptyset$. Then the sets $S_0 = S \cap C_0$ and $T_0 = T \cap C_0$ are disjoint stationary sets, with S_0 closed in S and T_0 closed in T. Therefore $S_0 \times T_0$ is a closed subspace of $S \times T$, so that if $S_0 \times T_0$ is not normal, then neither is $S \times T$. Therefore it is enough to consider the special case where $S \cap T = \emptyset$.

For each $\alpha \in S$, let $f(\alpha)$ be the first element of T that is greater than α. Because $\alpha < f(\alpha)$ for each $\alpha \in S$, we know from Corollary 2.9 that the set $C = \{\gamma < \omega_1 : \text{if } \alpha \in S \text{ and } \alpha < \gamma \text{ then } f(\alpha) < \gamma\}$ is a club-set in ω_1.

Because S is stationary, we know that $S \cap C$ is a relatively closed subset of S that is also stationary in ω_1 and that, by Corollary 2.9, if $\alpha < \alpha'$ are both in $S \cap C$, then $f(\alpha) < \alpha' < f(\alpha')$. Consider the set $F = \{\langle \alpha, f(\alpha) \rangle : \alpha \in S \cap C\} \subseteq S \times T$. If a sequence of points of F converges to a point $\langle \gamma, \delta \rangle \in S \times T$ then $\gamma = \delta$ and that is impossible because $S \cap T = \emptyset$. Therefore, no sequence of points of F can converge to any point of $S \times T$ so that every subset of F is closed in $S \times T$.

Use Corollary 2.8 to show that the stationary set $S \cap C$ is the union of two disjoint subsets S_1, S_2 that are also stationary. For $i = 1, 2$, let $F_i = \{\langle \alpha, f(\alpha) \rangle : \alpha \in S_i\}$. Then F_1 and F_2 are disjoint closed subsets of $S \times T$ so that, if $S \times T$ is normal, then there are open subsets U_i of $S \times T$ having $F_i \subseteq U_i$ and $cl(U_1) \cap cl(U_2) = \emptyset$ (where the closures are taken in $S \times T$).

Consider F_1 and U_1. For each $\alpha \in S_1$ the point $\langle \alpha, f(\alpha)\rangle \in U_1$ so that there is some $g_1(\alpha) < \alpha$ such that the set $(g_1(\alpha), \alpha]_S \times \{f(\alpha)\} \subseteq U_1$. Because S_1 is stationary, the Pressing Down Lemma gives some δ_1 with $g_1(\alpha) = \delta_1$ for every α in a stationary subset $S_1^* \subseteq S_1$ and therefore $(\delta_1, \alpha]_S \times \{f(\alpha)\} \subseteq U_1$ for every $\alpha \in S_1^*$. Similarly, there is some δ_2 and some stationary subset $S_2^* \subseteq S_2$ with $(\delta_2, \alpha]_S \times \{f(\alpha)\} \subseteq U_2$ for each $\alpha \in S_2^*$. Let $\delta = \max(\delta_1, \delta_2)$. Then for $i = 1, 2$, each $\alpha \in S_i^*$ has $(\delta, \alpha]_S \times \{f(\alpha)\} \subseteq U_i$. Let δ^+ be the first member of S with $\delta < \delta^+$.

For $i = 1, 2$, the sets $\{f(\alpha) : \alpha \in S_i^*\}$ are unbounded in ω_1 so that if D_i is the closure in ω_1 of the set $\{f(\alpha) : \alpha \in S_i^*\}$, then D_i is a club-set. Therefore, so is $D = D_1 \cap D_2$. Because the set T is stationary, there is some $\beta_0 \in D \cap T$.

We claim that the point $\langle \delta^+, \beta_0\rangle \in cl(U_1)$, where the closure is taken in $S \times T$. If $\beta_0 \in \{f(\alpha) : \alpha \in S_1^*\}$, say $\beta_0 = f(\alpha_0)$ where $\alpha_0 \in S_1^*$, then the point $\langle \delta^+, \beta_0\rangle$ is in the product set $(\delta, \alpha_0]_S \times \{f(\alpha_0)\} \subseteq U_1 \subseteq \mathrm{cl}(U_1)$. If $\beta_0 \notin \{f(\alpha) : \alpha \in S_1^*\}$, then there is a sequence $\alpha_n \in S_1^*$ with $\lim_{n\to\infty} f(\alpha_n) = \beta_0$. Then the points $\langle \delta^+, f(\alpha_n)\rangle$ lie in the sets $(\delta, \alpha_n]_S \times \{f(\alpha_n)\} \subseteq U_1$ so that $\langle \delta^+, \beta_0\rangle = \lim_{n\to\infty}\langle \delta^+, f(\alpha_n)\rangle \in \mathrm{cl}(U_1)$. Similarly, the point $\langle \delta^+, \beta_0\rangle \in \mathrm{cl}(U_2)$. But that is impossible because $\mathrm{cl}(U_1) \cap \mathrm{cl}(U_2) = \emptyset$. Therefore, the product space $S \times T$ is not normal when $S \cap T$ is not stationary. $\square$

We are now ready to prove the other half of Theorem 4.1. We present two proofs of the key lemma (4.3). The first is an elementary proof that involves repeated use of the Pressing Down Lemma and the theory of stationary sets outlines in Section 2 above, together with the fact that every metric space is normal. The second proof (see Remark 4.5, below) is much shorter and relies on some more specialized topics from general topology. Recall that if S and T are subsets of ω_1 and $\alpha \in \omega_1$, then $[0, \alpha]_T = [0, \alpha] \cap T$ and $[0, \alpha]_S$ is similarly defined.

Lemma 4.3 *Suppose S and T are subsets of ω_1. Suppose $\gamma < \omega_1$. Then the space $Y = S \times [0, \gamma]_T$ is normal.*

Proof: Note that $[0, \gamma]_T$ is countable and first-countable, and therefore is metrizable by Urysohn's theorem. By Proposition 3.1, the set S is either metrizable or stationary. If S is metrizable, then $S \times [0, \gamma]_T$ is also metrizable and therefore normal.

Now suppose that S is stationary, and suppose M and N are two disjoint closed subsets of $Y = S \times [0, \gamma]_T$ For each $\beta \in [0, \gamma]_T$, write $Horiz(\beta) = S \times \{\beta\}$. Let $Long(M) = \{\beta \in [0, \gamma]_T :$ $M \cap Horiz(\beta)$ is uncountable$\}$ and $Long(N) = \{\beta \in [0, \gamma]_T : Horiz(\beta) \cap N$ is uncountable$\}$. Because $Horiz(\beta)$ cannot contain two disjoint uncountable closed sets, $Long(M) \cap Long(N) = \emptyset$.

We claim that $Long(M)$ is a closed subset of $[0, \gamma]_T$. For suppose $\beta_n \in Long(M)$ and $\beta_n \to \beta^* \in [0, \gamma]_T$. Let $C_n = \{\alpha \in S : \langle \alpha, \beta_n\rangle \in M\}$. Each C_n is a relative club-set in S so that by Corollary 2.6 the diagonal intersection set $D := \bigcap\{C_n : n < \omega\}$ is also a relative club -set in S. For each $\alpha \in D$, $\langle \alpha, \beta_n\rangle \to \langle \alpha, \beta^*\rangle$ showing that $D \times \{\beta^*\} \subseteq M$ and hence that $\beta^* \in Long(M)$. Similarly $Long(N)$ is closed in $[0, \gamma]_T$. Because $[0, \gamma]_T$ is metrizable, there exist disjoint open sets $G, H \subseteq [0, \gamma]_T$ with $Long(M) \subseteq G$ and $Long(N) \subseteq H$.

As noted above, for each $\beta \in [0, \gamma]_T$, one or both of the sets $M \cap Horiz(\beta)$ and $N \cap Horiz(\beta)$ is countable so there is some $\alpha(\beta) < \omega_1$with the property that at least one one (or both) of the sets $M \cap Horiz(\beta)$ and $N \cap Horiz(\beta)$ is contained in $[0, \alpha(\beta)]_S \times \{\beta\}$. Because $[0, \gamma]_T$ is countable, the ordinal $\alpha^* = \sup\{\alpha(\beta) : \beta \in [0,\gamma]_T\} < \omega_1$. Now let $Y_1 = [0,\alpha^*]_S \times [0,\gamma]_T$ and $Y_2 = (\alpha^*, \omega_1)_S \times [0,\gamma]_T$. Then Y_1 and Y_2 are disjoint clopen subsets of Y and $Y = Y_1 \cup Y_2$.

The space Y_1 is metrizable by the Urysohn metrization theorem, so there are disjoint open subsets U_1, V_1 of Y_1 (and hence also of Y) with $M \cap Y_1 \subseteq U_1$ and $N \cap Y_1 \subseteq V_1$. Define $U_2 = (\alpha^*, \omega_1)_S \times G$ and $V_2 = (\alpha^*, \omega_1)_S \times H$ where G and H are the sets found above. Then U_2 and V_2 are open in Y_2 (and hence also in Y) and $U_2 \cap V_2 = \emptyset$. We claim that $M \cap Y_2 \subseteq U_2$. For suppose $\langle \alpha, \beta \rangle \in M \cap Y_2$. Then $\alpha > \alpha^* \geq \alpha(\beta)$ so that $M \cap Horiz(\beta)$ cannot be countable (because if it were countable, then $\alpha(\beta)$ would be an upper bound for its first coordinates). Therefore $\beta \in Long(M) \subseteq G$ so that $\langle \alpha, \beta \rangle \in U_2$. Similarly, $N \cap Y_2 \subseteq V_2$.

Let $U = U_1 \cup U_2$ and $V = V_1 \cup V_2$. Then $M \subseteq U$, $N \subseteq V$ and $U \cap V = \emptyset$, showing that Y is normal. □

Proposition 4.4 *If $S, T \subseteq \omega_1$ and $S \cap T$ is stationary, then the space $S \times T$ is normal.*

Proof: Let K and L be disjoint closed subsets of $S \times T$. For each $\alpha \in S \cap T$, the point $\langle \alpha, \alpha \rangle$ of $S \times T$ belongs to at most one of the sets K and L, so there is some $f(\alpha) < \alpha$ with the property that the set $O(\alpha) = (f(\alpha), \alpha]_S \times (f(\alpha), \alpha]_T$ is disjoint from at least one of the sets K and L. Then the Pressing Down Lemma (2.10) gives some γ such that the set $R := \{\alpha \in S \cap T : f(\alpha) = \gamma\}$ is stationary. Let $R_K = \{\alpha \in R : O(\alpha)$ is disjoint from $K\}$ and $R_L = \{\alpha \in R : O(\alpha)$ is disjoint from $L\}$. Because $R = R_K \cup R_L$, one of the sets R_K and R_L is stationary. Without loss of generality, suppose R_L is stationary. Then for each $\alpha \in R_L$ we see that $(\gamma, \alpha]_S \times (\gamma, \alpha]_T$ is disjoint from L so that $(\gamma, \omega_1)_S \times (\gamma, \omega_1)_T$ is disjoint from L.

Let $Y = S \times [0,\gamma]_T$ and $Z = [0,\gamma]_S \times (\gamma, \omega_1)_T$. By Lemma 4.3, the clopen subspace Y is normal, and an analogous proof shows that the clopen subspace Z is also normal. Therefore, we can find disjoint open subsets U_Y, V_Y of Y with $K \cap Y \subseteq U_Y$ and $L \cap Y \subseteq V_Y$, and we can find disjoint open subsets U_Z and V_Z of Z with $K \cap Z \subseteq U_Z$ and $L \cap Z \subseteq V_Z$. Because Y and Z are clopen in $S \times T$, the four sets U_T, V_Y, U_Z and V_Z are also open in $S \times T$. Now let $U = U_Y \cup U_Z \cup ((\gamma, \omega_1)_S \times (\gamma, \omega_1)_T)$ and $V = V_Y \cup V_Z$. Then $U \cap V = \emptyset$ and $K \subseteq U$ and $L \subseteq V$. Therefore $S \times T$ is normal. □

Remark 4.5 If one is willing to use theorems from general topology that do not usually appear in the first topology course, then there is a much shorter proof that the space $Y \times [0,\gamma]_T$ (and hence $S \times T$) is normal when $S \cap T$ is stationary. Induction shows that each set $[0,\gamma]$ embeds as a compact subset of the closed unit interval $I = [0,1]$, and the Pressing Down Lemma shows that that any stationary set S in ω_1 is countably paracompact. Then, because any stationary S is normal and countably paracompact, it follows that $S \times [0,1]$ is a normal space (see [5], Theorem 5.2.8). But then $S \times [0,\alpha^*]_T$, being an F_σ-subset of $S \times I$, is also normal. (see [5] Problem 2.1E, p. 73).

Corollary 4.6 *For any subspace $Y \subseteq \omega_1$, Y^2 is normal.*

Proof: If Y is stationary, apply Theorem 4.1 with $S = T = Y$. If Y is not stationary, apply Proposition 3.1 to see that Y is metrizable. But then Y^2 is also metrizable, and hence normal. □

Corollary 4.7 *For any subspace $Y \subseteq \omega_1$, if Y^2 is hereditarily normal (i.e., if every subspace of Y^2 is normal) then Y and Y^2 are metrizable.*

Proof: From 3.1 we know that Y is either metrizable or stationary. If Y is metrizable so is Y^2. If Y is stationary, then Corollary 2.8 gives disjoint stationary sets $S, T \subseteq Y$. Then $S \times T$ is a subspace of $Y \times Y$. But Proposition 4.2 shows that because $S \cap T = \emptyset$, the subspace $S \times T$ of Y^2 cannot be normal, so that Y^2 cannot be hereditarily normal. Therefore, we see that if Y^2 is hereditarily normal, then Y^2 is metrizable. □

5 Where to go for more

In this article we considered ω_1, its club-sets, and its stationary sets. These ideas generalize to any regular uncountable cardinal κ and can be found in most set theory books, e.g., [12]. The original study of homeomorphism between stationary sets is [4]. The basic study of products of stationary sets in κ is [11] in which Kemoto, Ohta, and Tamano characterize normality, and six related properties, in the product of two stationary subsets of κ. Buzyakova [2] proved that if S and T are stationary subsets of κ with $S \cap T = \emptyset$, then there is no continuous injection from the function space $C_p(S)$ into $C_p(T)$ where both function spaces carry the topology of pointwise convergence, so that in the light of Corollary 2.8 there are uncountably many topologically distinct types of function spaces associated with stationary subsets of ω_1.

Every linearly ordered set X with the open-interval topology is normal, but paracompactness is another matter entirely and, as proved in [6], a linearly ordered space fails to be paracompact if and only if it contains a closed set homeomorphic to a stationary set in an uncountable regular cardinal. That result was vastly generalized by Z. Balogh and M.E. Rudin who showed in [1] that a monotonically normal space[4] fails to be paracompact if and only if it contains a closed subset homeomorphic to a stationary set in an uncountable regular cardinal. That result is the key to the proof by Buzyakova and Vural that any monotonically normal topological group is paracompact [3]. Fleissner and Kunen [8] used disjoint stationary subsets $A, B \subseteq \omega_1$ to construct metrizable Baire spaces[5] X_A and X_B whose product is not a Baire space. In addition, stationary sets have been central tools in many other constructions in set-theoretic topology, and Fleissner's survey [7] and Rudin's monograph [16] are excellent places to look for details. A search in MathSciNet for "products of ordinals" will reveal many papers that investigate various normality and covering properties of (subspaces of) products of ordinals.

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[2] Buzyakova, R., Injections into function spaces over ordinals, *Topology and its Applications* 157(2010), 2844-2849.

[4] A space X is *monotonically normal* if for each pair (A, U) with A closed, U open, and $A \subseteq U \subseteq X$, there is an open set $G(A, U)$ with $A \subseteq G(A, U) \subseteq cl(G(A, U)) \subseteq U$ with the additional property that if $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$ have $A_1 \subseteq A_2$ and $U_1 \subseteq U_2$, then $G(A_1, U_1) \subseteq G(A_2, U_2)$. Every metric space, and every linearly ordered space is monotonically normal.

[5] A space X is a *Baire space* if the intersection of countably many dense open subsets of X is dense in X.

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