Subcompactness and Domain Representability in GO-spaces on Sets of Real Numbers

by

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Abstract: In this paper we explore a family of strong completeness properties in GO-spaces defined on sets of real numbers with the usual linear ordering. We show that if τ is any GO-topology on the real line $\mathbb{R}$, then $(\mathbb{R}, \tau)$ is subcompact, and so is any G_δ-subspace of $(\mathbb{R}, \tau)$. We also show that if (X, τ) is a subcompact GO-space constructed on a subset $X \subseteq \mathbb{R}$, then X is a G_δ-subset of any space $(\mathbb{R}, \sigma)$ where σ is any GO-topology on $\mathbb{R}$ with $\tau = \sigma|_X$. It follows that, for GO-spaces constructed on sets of real numbers, subcompactness is hereditary to G_δ-subsets. In addition, it follows that if (X, τ) is a subcompact GO-space constructed on any set of real numbers and if τ^S is the topology obtained from τ by isolating all points of a set $S \subseteq X$, then (X, τ^S) is also subcompact. Whether these two assertions hold for arbitrary subcompact spaces is not known.

We use our results on subcompactness to begin the study of other strong completeness properties in GO-spaces constructed on subsets of $\mathbb{R}$. For example, examples show that there are subcompact GO-spaces constructed on subsets $X \subseteq \mathbb{R}$ where X is not a G_δ-subset of the usual real line. However, if (X, τ) is a dense-in-itself GO-space constructed on some $X \subseteq \mathbb{R}$ and if (X, τ) is subcompact (or more generally domain-representable), then (X, τ) contains a dense subspace Y that is a G_δ-subspace of the usual real line. It follows that $(Y, \tau|_Y)$ is a dense subcompact subspace of (X, τ). Furthermore, for a dense-in-itself GO-space constructed on a set of real numbers, the existence of such a dense subspace Y of X is equivalent to pseudo-completeness of (X, τ) (in the sense of Oxtoby). These results eliminate many pathological sets of real numbers as potential counterexamples to the still-open question "Is there a domain-representable GO-space constructed on a subset of $\mathbb{R}$ that is not subcompact?" Finally, we use our subcompactness results to show that any co-compact GO-space constructed on a subset of $\mathbb{R}$ must be subcompact.

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1 Introduction

Subcompactness is one of the strong completeness properties, now called Amsterdam properties, that were introduced in the 1960s by de Groot and his collaborators. (See Section 2 for definitions.) Every complete metric space is subcompact, and every subcompact space is a Baire space (i.e., the intersection of countably many open, dense sets is dense). Furthermore, any product of subcompact

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spaces is subcompact (and hence Baire), something that contrasts sharply with behavior of Baire spaces in general.

It is somewhat surprising that, after 40 years, several of the most natural and important questions about subcompact spaces remain open. For example:

Q1) Suppose the topological space (X, τ) is subcompact and suppose that Y is a G_δ-subspace of X. Must $(Y, \tau|_Y)$ be subcompact?

Q2) Suppose (X, τ) is subcompact and $S \subseteq X$. Let τ^S be the topology on X having the collection $\tau \cup \{\{x\} : x \in S\}$ as a base. Must (X, τ^S) be subcompact?

In recent years, another completeness property called *domain representability* has attracted topologists' attention. Domain representability is a strong completeness property in the sense that any product of domain-representable spaces is domain representable and hence a Baire space. The analogs of Q1 and Q2 for domain-representable spaces are known to have affirmative answers (see [3], [4]). It is known that domain representability lies somewhere between subcompactness and strong Choquet completeness: every subcompact T_3-space is domain representable [4], and every domain representable space is strongly Choquet complete [9]. The second of those two implications is irreversible, but the next question remains open:

Q3) Does domain representability imply subcompactness?

We expect a negative answer to Q3 among general spaces. However, to date, progress on Q3 has been limited to showing that subcompactness and domain representability are equivalent properties in certain restricted classes of spaces, e.g., in metrizable spaces [9], in Moore spaces and in BCO spaces [6], and in function spaces with the pointwise convergence topology [5]. Note that in any class of spaces where subcompactness and domain representability are equivalent concepts, questions Q1 and Q2 have affirmative answers.

In this paper, $\mathbb{R}$ denotes the set of real numbers with the usual ordering and we focus on the class of generalized ordered spaces constructed on subsets of $\mathbb{R}$. Generalized ordered spaces (GO-spaces) constructed on $\mathbb{R}$ and its subspaces have been important examples in topology – for example, the Sorgenfrey and Michael lines and their subspaces have been used to study product spaces. These GO-spaces have also been used in the study of the Amsterdam completeness properties: in [1], it was shown that the Sorgenfrey line is co-compact and subcompact, but no dense subspace of it can be base-compact or regularly co-compact.

The goal of this paper is to study Questions Q1, Q2 and Q3 in the category of GO-spaces constructed on sets of real numbers. We answer both Q1 and Q2 affirmatively for GO-spaces on sets of real numbers, and make some progress on Q3, showing that if $X \subseteq \mathbb{R}$ and if (X, τ) is a domain-representable, dense-in-itself GO-space (with respect to the usual ordering), then (X, τ) contains a subcompact, dense G_δ-subspace Y. In addition, Y is a G_δ-subset of the usual open-interval topology on $\mathbb{R}$ and this eliminates many subsets of $\mathbb{R}$ as potential counterexamples to Q3. However, we do not know whether every domain-representable GO-space defined on a set of real numbers must be subcompact so that these GO-spaces remain a potential source for counterexamples to Q3. In addition, we use our results on subcompactness to study the role of Oxtoby's pseudo-completeness in GO-spaces constructed on subspaces of $\mathbb{R}$.

Necessary definitions appear in Section 2 of our paper. Our main results on subcompactness appear in Section 3, and in Section 4 we use these results to study other strong completeness properties in GO-spaces constructed on subsets of $\mathbb{R}$. In Section 5 we pose several open questions about strong completeness properties of GO-spaces defined on sets of real numbers.

2 Basic definitions

The four basic Amsterdam properties are co-compactness, regular co-compactness, base compactness, and subcompactness. A regular space X is *co-compact* if there is a collection $\mathcal{C}$ of closed subsets of X such that any centered subcollection[3] of $\mathcal{C}$ has nonempty intersection, and such that if $p \in U$ with U open, then some $C \in \mathcal{C}$ has $p \in \mathrm{Int}(C) \subseteq C \subseteq U$. If members of $\mathcal{C}$ are regularly closed sets, i.e., if each is the closure of its own interior, then X is *regularly co-compact.* The space X is *base compact* if there is a base $\mathcal{B}$ of nonempty, open sets with the property that $\bigcap\{\mathrm{cl}(C) : C \in \mathcal{C}\} \neq \emptyset$ whenever $\mathcal{C}$ is a centered subcollection of $\mathcal{B}$. Finally, a space X is *subcompact* if it has a base $\mathcal{B}$ of nonempty open sets such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ has the property that given any $F_1, F_2 \in \mathcal{F}$, some $F_3 \in \mathcal{F}$ has $\mathrm{cl}(F_3) \subseteq F_1 \cap F_2$. Such an $\mathcal{F}$ is said to be a *regular filter base* in $\mathcal{B}$, and the base $\mathcal{B}$ is said to be a *subcompact base* for X. It is clear that regular co-compactness implies base compactness, and base compactness implies subcompactness. (As the example of the Sorgenfrey line shows, the fourth property, co-compactness, is strictly weaker than regular co-compactness and does not imply base compactness.) Most of this paper will focus on subcompact spaces, but questions in the final section involve other Amsterdam properties as well.

Another classical completeness property is pseudo-completeness, introduced by Oxtoby in [12]. A regular space is *pseudo-complete* if there is a sequence of π-bases $\mathcal{P}(n)$ such that if $P_n \in \mathcal{P}(n)$ and $\mathrm{cl}(P_{n+1}) \subseteq P_n$ for each $n \geq 1$, then $\bigcap\{P_n : n \geq 1\} \neq \emptyset$.

As mentioned in the Introduction, there is a newer topological completeness property called domain-representability that was borrowed from theoretical computer science. Defining the property requires certain background information. Let $(P, \sqsubseteq)$ be a partially ordered set. By the supremum of a subset $S \subseteq P$ we mean an upper bound u for S that has $u \sqsubseteq v$ whenever v is an upper bound for S. A subset $S \subseteq P$ is *directed* if it is nonempty and has the property that given any $s_1, s_2 \in S$, some $s_3 \in S$ has $s_1, s_2 \sqsubseteq s_3$. If every directed subset of P has a supremum in P, then P is a *dcpo* (= directed complete partial order). Zorn's lemma shows that for every p in a dcpo P, there is some maximal element $q \in P$ with $p \sqsubseteq q$. The set of all maximal elements of P is denoted by $\max(P)$.

There is an important auxiliary relation $\ll$ on P defined as follows: we say that $a \ll b$ if whenever a directed set $S \subseteq P$ has $b \sqsubseteq \sup(S)$, then some $s \in S$ has $a \sqsubseteq s$. The poset P is *continuous* if for each $a \in P$ the set $\Downarrow(a) := \{b \in P : b \ll a\}$ is directed and has a as its supremum. A continuous dcpo is called a *domain.*

In a domain P, the collection of all sets $\Uparrow(a) := \{b \in P : a \ll b\}$ is a base for a topology known as the *Scott topology* on P. The set $\max(P)$ is a dense subset of P in the Scott topology. If for a topological space X there is a domain $(P, \sqsubseteq)$ such that X is homeomorphic to $\max(P)$ with the relative Scott topology, then we say that X is *domain representable.*

[3] A collection $\mathcal{D}$ is *centered* if it has the finite intersection property.

Recall that a *generalized ordered space* (= GO-space) is a triple $(X, <, \tau)$ where $(X, <)$ is a linearly ordered set and τ is a T_2-topology on X that has a base of order-convex sets. The open-interval topology of the ordering $<$, which we will always denote by λ, is the most familiar *GO-topology*. If $X \subseteq \mathbb{R}$ and $<$ is the usual linear ordering of $\mathbb{R}$, then any GO-space $(X, < |X, \tau)$ is said to be a *GO-space constructed on a subset of* $\mathbb{R}$. The most familiar GO-spaces constructed on $\mathbb{R}$ are the Sorgenfrey and Michael lines.

3 Subcompactness in GO-spaces constructed on sets of real numbers

Because the real line $(\mathbb{R}, \lambda)$ with its usual open interval topology is hereditarily Lindelöf, given any subset $S \subseteq \mathbb{R}$ there is a countable subset $S_0 \subseteq S$ such that for every $x \in S - S_0$ and every $\epsilon > 0$, both $(x - \epsilon, x) \cap S$ and $(x, x + \epsilon) \cap S$ are uncountable. We will say that such a point x is a *two-sided condensation point* of S. Because $(\mathbb{R}, \lambda)$ is also hereditarily separable, we may assume that the countable set S_0 is dense in S. Write $S_1 = S - S_0$.

We begin with a technical lemma about certain G_δ-subspaces of $(\mathbb{R}, \lambda)$. Any such subspace is a completely metrizable, so that we can invoke ideas from Baire Category theory provided we are careful to use relativized versions of those ideas. For example, the usual Cantor set is not the union of countably many relatively nowhere dense subspaces of itself, even though the usual Cantor set is nowhere dense in $\mathbb{R}$. It is well known that any uncountable G_δ-subset Y of $(\mathbb{R}, \lambda)$ contains a Cantor set (= a compact, uncountable, dense-in-itself, totally disconnected subset). We need a little more in our next lemma.

Lemma 3.1 *Let Z be a closed uncountable subset of $(\mathbb{R}, \lambda)$ and let $Z = Z_0 \cup Z_1$ be as above. Suppose that I is a convex subset of $\mathbb{R}$ such that the set $Y := I \cap Z_1$ is uncountable. Suppose that $C = \bigcup\{C_k : k \geq 1\}$ where each C_k is a relatively closed, relatively nowhere dense subset of $(Z_1, \lambda|_{Z_1})$. Let $s = \inf(I \cap Z_1)$ and $t = \sup(I \cap Z_1)$. Then $s < t$ and there are sets A and B such that*

a) $A \cup B \subseteq Z_1$, with $s = \inf(A),\ t = \sup(B)$,

b) $A - \{s\} \subseteq (s, \frac{s+t}{2}] \cap Y$,

c) $B - \{t\} \subseteq [\frac{s+t}{2}, t) \cap Y$, and

d) $C \cap (A \cup B) \subseteq \{s, t\}$,

e) $A \cup B$ is relatively nowhere dense in Z_1.

In addition, there are strictly increasing functions $\alpha : (s, t] \to A - \{s\}$ and $\beta : [s, t) \to B - \{t\}$ such that $\alpha(y) < y$ for each $y \in (s, t]$ and $y < \beta(y)$ for each $y \in [s, t)$.

Proof: Let I^o denote the λ-interior of I. Then $I^o \cap Z_1$ is a nonempty open subset of Z_1 so that $(I^o \cap Z_1) - C$ is a dense G_δ-subset of $I^o \cap Z_1$ and hence $(I \cap Z_1) - C$ is a dense G_δ-subset of $I \cap Z_1$.

Write $Y = (I \cap Z_1) - C$. Then $\inf(Y) = s$ and $\sup(Y) = t$. In addition, Y for each $\epsilon > 0$ both $[s, s+\epsilon) \cap Y$ and $(t-\epsilon, t] \cap Y$ are uncountable. Therefore, standard techniques provide Cantor sets A^+ and B^+ with

1) $s \in A^+ \subseteq [s, \frac{s+t}{2}),\ t \in B^+ \subseteq (\frac{s+t}{2}, t]$,

2) $A^+ - \{s\} \subseteq Y,\ B^+ - \{t\} \subseteq Y$, and

3) the sets $A = A^+ \cap Y$ and $B = B^+ \cap Y$ are relatively nowhere dense in Z_1

In order to obtain b) and c) of the lemma, we may replace A by $[s, \frac{s+t}{2}] \cap A$ and B by $B \cap [\frac{s+t}{2}, t]$

Next we define the function β. Let $S := \{s_k : k \geq 1\}$ be any countable dense set in $[s, t)$ in the λ-topology, with $s_1 = s$. Using the fact that all but countably many points of B are two-sided limit points of B^+, we recursively find two-sided limit points $d_k \in B^+$ with the following properties:

i) if $s_i < s_j$ then $d_i < d_j$

ii) for each $i \geq 1,\ s_i < \frac{s_i+t}{2} < d_i$

For any $y \in (s, t)$ define $\beta(y) = \sup\{d_i : s_i \leq y\}$. Because B^+ is compact and each $d_i \in B^+$, we see that each $\beta(y) \in B$. Next we show that β is a strictly increasing function. Suppose $y < y'$ in $[s, t)$. Find $s_j, s_k \in (y, y') \cap S$ with $s_j < s_k$. Then $d_j < d_k$ and for any $s_i \leq y$ we have $d_i < d_j < d_k$. Hence $\beta(y) \leq d_j$ and $d_k \leq \beta(y')$ so that $\beta(y) < \beta(y')$ as claimed. Because β is strictly increasing, we know that $\beta(y) \in B^+ - \{t\} \subseteq B$ for each $y \in [s, t)$. Hence $\beta(y) \in Y - C$. Finally, we show that $y < \beta(y)$ for each $y \in [s, t)$. Fix any such y. If $s_i \leq y$ then $\frac{s_i+t}{2} < d_i \leq \beta(y)$ so that

$$y = \sup\{s_i : s_i \leq y\} \leq \sup\{\frac{s_i + t}{2} : s_i \leq y\} \leq \sup\{d_i : s_i \leq y\} = \beta(y).$$

But $\sup\{\frac{s_i+t}{2} : s_i \leq y\} = \frac{y+t}{2}$ so we have $y < \frac{y+t}{2} \leq \beta(y)$ as claimed. □

Our next result concerns GO-spaces constructed on the entire set of real numbers. It may be proved directly, by a generalization of the recursive technique used in [1] to show that the Sorgenfrey line is subcompact. However, it is an immediate corollary of the more general Theorem 3.3, which we prove in detail below.

Proposition 3.2 *Suppose that τ is any GO-topology on the linearly ordered set $(\mathbb{R}, <)$. Then $(\mathbb{R}, \tau)$ is subcompact.* □

The key idea in the direct proof of Proposition 3.2 involves finding base elements that never repeat certain kinds of endpoints called "external endpoints," and that idea is also the key to proving our next result. It is surprising how much harder the proof becomes when G_δ-subsets of $\mathbb{R}$, rather than all of $\mathbb{R}$, are involved.

Theorem 3.3 *Suppose that τ is a GO-topology on $\mathbb{R}$ and that X is a G_δ-subset of the space $(\mathbb{R}, \tau)$. Then $(X, \tau|_X)$ is subcompact.*

Proof: Write $X = X_0 \cup X_1$ where X_0 is countable and dense in (X, λ_X) and where each point of $X_1 = X - X_0$ is a two-sided condensation point of X_1. Our proof will have three main steps. In the first we define basic open τ_X-neighborhoods of the countably many points in X_0. In the second, if $X_1 \neq \emptyset$, we define basic τ_X-neighborhoods of the uncountably many points in X_1. In the third step we show that the collection of basic neighborhoods found in the first two steps is a subcompact base for (X, τ).

Step 1: Find basic neighborhoods for points of X_0 Write $X = \bigcap\{H(n) : n \geq 1\}$ where $H(n) \in \tau$ and $H(n+1) \subseteq H(n)$ for each $n \geq 1$. Without repetitions, index the set $X_0 := \{x_k : k \geq 1\}$. We will recursively define a collection $\{B(x_j, k) : 1 \leq j \leq k < \omega\}$ of intervals in $(\mathbb{R}, <)$ in the following pattern: first $B(x_1, 1)$, then $B(x_1, 2)$ and $B(x_2, 2)$, then $B(x_1, 3), B(x_2, 3), B(x_3, 3)$ and so on. We need to classify the points of X based on the "shape" of their $\tau|_X$-neighborhoods. Let

$I = \{x \in X : \{x\} \in \tau|_X\}$
$R = \{x \in X - I : X \cap [x, \rightarrow) \in \tau|_X\}$
$L = \{x \in X - I : X \cap (\leftarrow, x] \in \tau|_X\}$
$E = X - (I \cup R \cup L)$.

By an *external endpoint* of one of the intervals $B(x, k)$ we will mean any endpoint of $B(x, k)$ except for the point x. Thus $\mathrm{cl}_\lambda(B(x, k)) - B(x, k)$ is the set of external endpoints of $B(x, k)$.

To initialize the recursion in Step 1, consider x_1. If $x_1 \in I$, let $B(x_1, 1) := \{x_1\}$. If $x_1 \in R$, choose $b(x_1, 1)$ in the infinite set $(x_1, x_1+1) \cap X$ and so that the set $B(x_1, 1) := [x_1, b(x_1, 1)) \subseteq [x_1, b(x_1, 1)] \subseteq H(2)$. If $x_1 \in L$, let $B(x_1, 1) := (a(x_1, 1), x_1] \subseteq [a(x_1, 1), x_1] \subseteq H(2)$ where $a(x_1, 1)$ is a point of $(x_1 - 1, x_1) \cap X$. Finally if $x_1 \in E$ let $B(x_1, 1) = (a(x_1, 1), b(x_1, 1))$ where $[a(x_1, 1), b(x_1, 1)] \subseteq H(2)$ with $a(x_1, 1), b(x_1, 1)$ chosen as above. Let $S(x_1, 1) = B(x_1, 1) \cap X$ and let $E(1)$ be the set of external endpoints of $B(x_1, 1)$, i.e., $E(1) = \{\inf(B(x_1, 1)), \sup(B(x_1, 1))\} - \{x_1\}$.

To describe the general recursion step, suppose we have $x_i \in B(x_i, j) \subseteq \mathrm{cl}_\lambda(B(x_i, j)) \subseteq H(i+j)$ for $1 \leq i \leq j \leq n$ and that $E(n)$ is the finite set of external endpoints of the sets $B(x_i, j)$ for $1 \leq i \leq j \leq n$. Choose $\epsilon > 0$ so small $\epsilon < \frac{1}{n+1}$ and such that for $1 \leq i \leq n+1$, $[x_i - \epsilon, x_i + \epsilon] \subseteq H(i+n+1)$, and such that $(x_i - \epsilon, x_i) \cup (x_i, x_i + \epsilon)$ contains no point of the finite set $E(n)$. (We cannot ask that $(x_i - \epsilon, x_i + \epsilon) \cap E(n) = \emptyset$ because x_{n+1} might have been an external endpoint of some previously defined $B(x_i, j)$.) There are four possibilities for $B(x_1, n+1)$. If $x_1 \in I$, let $B(x_1, n+1) = \{x_1\}$. If $x_1 \in R$, then the set $X \cap (x_1, x_1 + \epsilon)$ is infinite and we choose $b(x_1, n+1) \in (x_1, x_1 + \epsilon) - E(n)$. Letting $B(x_1, n+1) = [x_1, b(x_1, n+1))$, we know that the external end point $b(x_1, n+1) \notin E(n)$. The cases where $x_1 \in L$ and $x_1 \in E$ are handled analogously. Let $E(n, 1)$ be the union of $E(n)$ with the set of external endpoints of $B(x_1, n+1)$. Next consider x_2. Shrinking ϵ if necessary, we may assume that $(x_2 - \epsilon, x_2) \cup (x_2, x_2 + \epsilon)$ has no points in common with $E(n, 1)$ and this allows us to define the sets $B(x_2, n+1)$ with four cases (depending upon which of the sets I, R, L and E contains x_2). Let $E(n, 2)$ be the union of $E(n, 1)$ with the set of external endpoints of the set $B(x_2, n+1)$. Repeating this process $n+1$ times gives the sets $B(x_i, n+1)$ for $1 \leq i \leq n+1$ and the finite set $E(n+1)$, which is the set of all external endpoints of all sets $B(x_i, j)$ for $1 \leq i \leq j \leq n+1$. Note that $\mathrm{cl}_\lambda(B(x_i, n+1)) \subseteq H(i+n+1)$ for $1 \leq i \leq n+1$.

Write $S(x_k, j) = X \cap B(x_k, j)$. Then the collection $\{S(x_i, j) : j \geq i\}$ is a local base at x_i in the space (X, τ). We claim:

> Claim 1: if $\langle S(x_{i_n}, j_n) : n \geq 1\rangle$ is a strictly decreasing sequence of the sets defined in Step 1 above, then $\bigcap\{S(x_{i_n}, j_n) : n \geq 1\} \neq \emptyset$.

To verify Claim 1, suppose the intersection is empty. Because the sets $S(x_{i_n}, j_n)$ are nested and $S(x_{i_n}, j_n) \subseteq B(x_{i_n}, j_n)$, the collection $\{B(x_{i_n}, j_n) : n \geq 1\}$ is a centered collection of bounded intervals in $\mathbb{R}$, and therefore there is some point $z \in \bigcap\{\mathrm{cl}_\lambda(B(x_{i_n}, j_n)) : n \geq 1\}$. Because the intersection of the sets $S(x_{i_n}, j_n)$ is empty, z fails to belong to one of those sets, say $S(x_{i_m}, j_m)$, and therefore fails to belong to all sets $S(x_{i_k}, j_k)$ for $k \geq m$. Without loss of generality, suppose $m = 1$.

Because the sets $S(x_{i_n}, j_n)$ are distinct, we must have no repetitions in the ordered pairs (x_{i_n}, j_n) naming the sets. Consequently, the sequence of sums $i_n + j_n$ must be unbounded so that $z \in \mathrm{cl}_\lambda(B(x_{i_n}, j_n)) \subseteq H(i_n + j_n)$. Because the sets $H(k)$ are nested and have $X = \bigcap\{H(k) : k \geq 1\}$, we know that $z \in X$. Then $z \notin S(x_{i_n}, j_n) = B(x_{i_n}, j_n) \cap X$ combines with $z \in X \cap \mathrm{cl}_\lambda(B(x_{i_n}, j_n))$ to show that z must be an endpoint of each interval $B(x_{i_n}, j_n)$. Because each $x_{i_n} \in S(x_{i_n}, j_n)$ while $z \notin S(x_{i_n}, j_n)$ we see that z must be an external endpoint of $B(x_{i_n}, j_n)$ for each n and that is impossible because the sets $B(x_{i_n}, j_n)$ cannot repeat any external endpoint. Therefore, Claim 1 is established.

<u>Step 2: Construct basic neighborhoods for the points of X_1</u>: Suppose $X_1 \neq \emptyset$. We know that the set $X_1 = X - X_0$ is a G_δ-subset of $(\mathbb{R}, \tau)$, is uncountable, and every point of X_1 is a two-sided condensation point of X_1. Write $X_1 = \bigcap\{G_n : n \geq 1\}$ where $G_n \in \tau$ and $G_{n+1} \subseteq G_n$. Write each G_n as the union of its convex components in the linearly ordered set $(\mathbb{R}, <)$, say $G_n = \bigcup\{G(n, j) : j \in J(n)\}$. Because we can replace the space $(\mathbb{R}, \tau)$ by a homeomorphic copy of itself in $(0, 1)$, there is no loss of generality if we assume that each $G(n, j)$ has finite diameter with respect to the usual metric on $\mathbb{R}$. In addition, because the set X_0 is dense in X, if $x \in X_1$ and if $G(n, j_n)$ is the unique convex component of G_n that contains x, we know that the limit (as $n \to \infty$) of the diameter of $G(n, j_n)$ must be zero. Because there may be may isolated points in $\tau|X$, the index sets $J(n)$ might be uncountable. Let $J(n, 1) := \{j \in J(n) : |G(n, j) \cap X_1| > 1\}$. For each $j \in J(n, 1)$ the set $G(n, j) \cap X_1$ is uncountable, and the index set $J(n, 1)$ is countable.

Let $Z = \mathrm{cl}_\lambda(X_1)$ and partition $Z = Z_0 \cup Z_1$ as described above. Note that Z is a G_δ-subset of $(\mathbb{R}, \lambda)$ and hence so is Z_1. Also note that $X_1 \subseteq Z_1$.

<u>Step 2, Level $n = 1$</u>: Fix $j \in J(1, 1)$. Then the set $G(1, j) \cap X_1$ is uncountable and hence so is $Z_1 \cap G(1, j)$. Apply Lemma 3.1 with $G(1, j)$ being the set called I in (3.1), and $C = \emptyset$. Compute $s = \inf(Z_1 \cap G(1, j))$ and $t = \sup(Z_1 \cap G(1, j))$, and find relatively closed, relatively nowhere dense sets $A = A(1, j), B = B(1, j)$ with $s \in A \subseteq \{s\} \cup (Z_1 \cap G(1, j))$ and $t \in B \subseteq (Z_1 \cap G(1, j)) \cup \{t\}$. Also find strictly increasing functions $\alpha = \alpha_{1,j} : (s, t] \to A(1, j) - \{s\}$ and $\beta = \beta_{1,j} : [s, t) \to B(1, j) - \{t\}$ with $\alpha(x) < x$ for $x \in (s, t]$ and $x < \beta(x)$ for each $x \in [s, t)$.

Now consider any $x \in X_1 \cap G(1, j)$. If $x \in I$ (see above for the definitions of I, R, L and E), let $B(x, 1) = \{x\}$. If $x \in R$, we would like to define $B(x, 1) = [x, \beta(x))$, but in order to do that we must show that $x < t$. We establish $x < t$ by contradiction. Clearly $x \leq t$, so for contradiction, assume $x = t$. Then we claim that $x = t = \sup(G(1, j))$. For otherwise, $t < \sup(G(1, j))$ so that we may find $\epsilon > 0$ with $[x, x + \epsilon) \subseteq G(1, j)$. Because $x \in X_1$, the point x is a two-sided condensation point of X_1 so that $X_1 \cap (x, x + \epsilon)$ is uncountable. Hence $t = x$ is not the supremum of $X_1 \cap G(1, j)$, which is false. Therefore, we conclude that if $x = t$ then $x = t = \sup(G(1, j))$. But $t = x \in X_1 \cap G(1, j)$ so that the set $G(1, j)$ contains its own right endpoint, namely x. Then $x \in G(1, j) \in \tau$ so that $x \in R$ gives us some $\delta > 0$ with $[x, x + \delta) \subset G(1, j)$, contrary to $x = \sup(G(1, j))$. This cluster of contradictions shows that $x = t$ is impossible, as claimed. Now we may define $B(x, 1) = [x, \beta(x))$ if $x \in R \cap X_1 \cap G(1, j)$. Similarly, if $x \in L \cap X_1 \cap G(1, j)$ then

$s < x$ and we may define $B(x,1) = (\alpha(x), x]$ and if $x \in E \cap X_1 \cap G(1,j)$ then $s < x < t$ and we may define $B(x,1) = (\alpha(x), \beta(x))$. In any case, note that $\mathrm{cl}_\lambda(B(x,1)) \subseteq G(1,j) \subseteq G(1)$.

The term *external endpoint* is used just as in Step 1 of the proof. Note that for $x \in X_1 \cap G(1,j)$ every external endpoint of $B(x,1)$ is a point of $A(1,j) \cup B(1,j)$ which is a relatively closed, relatively nowhere dense subset of Z_1. Also note that the points $s = \inf(Z_1 \cap G(1,j))$ and $t = \sup(Z_1 \cap G(1,j))$ are never external endpoints of any set $B(x,1)$ for $x \in X_1 \cap G(1,j)$.

Because any point $x \in X_1$ belongs to a unique $G(1,j)$ we have now defined $B(x,1)$ for each $x \in X_1$. We let $End(1)$ be the set of all external endpoints of all sets $B(x,1)$ for $x \in X_1$. Then $End(1)$ is a relative first category subset of Z_1 because the index set $J(1,1)$ is countable and $End(1) \subseteq \bigcup\{A(1,j) \cup B(1,j) : j \in J(1,1)\}$.

> Claim 2: Suppose $x \neq x'$ are points of X_1 and that $B(x,1) \cap X \subseteq B(x',1)$. It cannot happen that both x and x' belong to a single one of the sets I, R, L or E.

In proving Claim 2, there is no loss of generality if we assume $x < x'$. Let $j, j' \in J(1)$ with $x \in G(1,j), x' \in G(1,j')$. If $j \neq j'$ then $B(x,1) \cap B(x',1) \subseteq G(1,j) \cap G(1,j') = \emptyset$, contrary to $x \in B(x,1) \cap X \subseteq B(x',1)$, so that $j = j'$. Clearly $x, x' \in I$ is impossible. Given that $x \notin I$ and $x \in G(1,j)$ we conclude that because $G(1,j) \cap X_1$ is a relative τ-neighborhood of x, it must contain an open interval on the left or right of x, so that because points of X_1 are two-sided condensation points of X_1, the set $G(1,j) \cap X_1$ must be uncountable. Hence we have sets $A_{1,j}, B_{1,j}$ and functions $\beta = \beta_{1,j}$ and $\alpha = \alpha_{1,j}$. Now consider the case where $x, x' \in R$. Then $B(x,1) = [x, \beta(x))$ and $B(x',1) = [x', \beta(x'))$. From $x \in B(x,1) \cap X \subseteq B(x',1)$ we get $x' \leq x < \beta(x')$ which is false because $x < x'$. Hence $\{x, x'\} \subseteq R$ is impossible. Next consider the case where $x, x' \in L$. Then $x < x'$ gives $\alpha(x) < \alpha(x')$. Now the point $\alpha(x) \in Z_1$ so that the interval $(\alpha(x), \alpha(x'))$ must contain uncountably many points of Z_1. Choose any $z_1 \in (\alpha(x), \alpha(x')) \cap Z_1$. Because Z is the λ-closure of X_1 and $z_1 \in Z$, there is a point $x_1 \in X_1 \cap (\alpha(x), \alpha(x'))$. Clearly, then, $x_1 \notin B(x',1)$. Recall that $x \in B(x,1) \cap X \subseteq B(x',1) = (\alpha(x'), x']$ so that we have $\alpha(x') < x < x'$. Consequently, $\alpha(x) < x_1 < \alpha(x') < x$ showing that $x_1 \in (\alpha(x), x] \cap X = B(x,1) \cap X \subseteq B(x',1)$, which is impossible. Thus, $x, x' \in L$ is also impossible. As the final step in proving Claim 1, consider the case where $x, x' \in E$. Then $B(x,1) = (\alpha(x), \beta(x))$ and $B(x',1) = (\alpha(x'), \beta(x'))$ so that from $x \in B(x,1) \cap X \subseteq B(x',1)$ we have $\alpha(x') < x < \beta(x')$. From $x < x'$ we know that $\alpha(x) < \alpha(x')$. Therefore $\alpha(x) < \alpha(x') < x$. But $\alpha(x) \in Z_1$ so that the interval $(\alpha(x), \alpha(x'))$ must contain uncountably many points of Z_1. Then the interval $(\alpha(x), \alpha(x'))$ must contain some point $x_1 \in X_1$ because Z is the λ-closure of X_1. Because $x_1 < \alpha(x')$ we know that $x_1 \notin B(x',1)$ and yet from $\alpha(x) < x_1 < \alpha(x') < x$ we know that $x_1 \in B(x,1) \cap X$, which again contradicts $B(x,1) \cap X \subseteq B(x',1)$. Consequently, Claim 2 is established.

> Claim 3: For $x \in X_1$ let $S(x,1) = B(x,1) \cap X$. It is not possible to have an infinite sequence $x_i \in X_1$ of distinct points such that $S(x_{i+1},1) \subseteq S(x_i,1)$ for each $i \geq 1$.

For contradiction, suppose there were an infinite sequence of distinct points $x_i \in X_1$ with $S(x_{i+1},1) \subseteq S(x_i,1)$. If some $x_i \in I$ then $S(x_i,1) = \{x_i\}$ so that $x_{i+1} \in S(x_{i+1},1) \subseteq S(x_i,1) = \{x_i\}$ yields $x_{i+1} = x_i$, and that is false. Hence $x_i \in R \cup L \cup E$ for all i. Suppose $x_1 \in R$. Then Claim 2 shows that $x_i \notin R$ for each $i \geq 2$, so that $x_2 \in L \cup E$. If $x_2 \in L$, then Claim 2 shows that $x_i \notin L \cup R$

for each $i \geq 3$, so $x_3 \in E$. But then Claim 2 shows that $x_4 \notin I \cup R \cup L \cup E = X$, and that is false. All other cases are similar, so Claim 3 is proved.

Step 2, Level $n+1$: Suppose that $B(x, j)$ is defined for each $x \in X_1$ and each $j \leq n$ and that the set $End(n)$ of all external endpoints of all of the previously defined sets $B(x, j)$ is known to be a countable union of relatively closed, relatively nowhere dense subsets of Z_1. We define $B(x, n+1)$ by making cosmetic changes in the $n = 1$ step above.

For a fixed $x \in X_1$, choose the unique $j = j(n+1, x) \in J(n+1)$ with $x \in G(n+1, j)$. If $x \in I$, let $B(x, n+1) = \{x\}$. If $x \notin I$, then $j \in J(n+1, 1)$ and the set $G(n+1, j) \cap X_1$ is uncountable. Because $X_1 \subseteq Z_1$, we know that the set $G(n+1, j) \cap Z_1$ is uncountable. Compute $s = \inf(G(n+1, j) \cap Z_1)$ and $t = \sup(G(n+1, j) \cap Z_1)$, and apply Lemma 3.1 with $I = G(n+1, j)$ and $C = End(n)$ to find relatively closed, relatively nowhere dense subsets $A(n+1, j), B(n+1, j)$ of Z_1 and functions $\alpha_{n+1,j} : (s, t] \to A(n+1, j) - \{s\}, \beta_{n+1,j} : [s, t) \to B(n+1, j) - \{t\}$. If $x \in R$, then, as in the case where $n = 1$, we prove that $x < t$ and we let $B(x, n+1) = [x, \beta_{n+1,j}(x))$. If $x \in L$, then $x > s$ and we let $B(x, n+1) = (\alpha_{n+1,j}(x), x]$. If $x \in E$ then $s < x < t$ and we let $B(x, n+1) = (\alpha_{n+1,j}(x), \beta_{n+1,j}(x))$. Note that for any $x \in X_1 \cap G(n+1, j)$ we have $\mathrm{cl}_\lambda(B(x, n+1)) \subseteq G(n+1, j)$.

Still considering the fixed $x \in X_1$ and $j = j(n = 1, x)$ from the previous paragraph, we claim that no external endpoint of $B(x, n+1)$ can belong to the set $End(n)$. Clearly, if $x \in I$, then $B(x, n+1)$ has no external endpoints. In case $x \in R$, then $\beta_{n+1,j}(x)$ is the only external endpoint of $B(x, n+1)$ and we note that $\beta_{n+1,j}(x) \in B(n+1, j) - \{t\} \subseteq Y - C = Y - End(n)$ where $s = \inf(Z_1 \cap G(n+1, j)$ and $t = \sup(Z_1 \cap G(n+1, j)$ as computed in the previous paragraph. The case where $x \in L$ is analogous, and in case $x \in E$ then the set of external endpoints of $B(x, n+1)$ is $\{\alpha_{n+1,j}(x), \beta_{n+1,j}(x)\}$ which is disjoint from $End(n)$. This argument shows that no external endpoint of $B(x, n+1)$ can be a repetition of an external endpoint from a previous level.

The above process defines $B(x, n+1)$ for each $x \in X_1$. Let $End(n+1)$ be the union of $End(n)$ with the collection of all external endpoints of all sets $B(x, n+1)$ for $x \in X_1$. As before, $End(n+1)$ is a subset of a countable union of relatively closed, relatively nowhere dense subsets of Z_1 because the index set $J(n+1, 1)$ is countable, and the very same proofs used for Claims 2 and 3 give us

> Claim 4: For $x \in X_1$, let $S(x, n+1) = B(x, n+1) \cap X$. It is not possible to have an infinite sequence $x_i \in X_1$ of distinct points such that $S(x_{i+1}, n+1) \subseteq S(x_i, n+1)$ for each $i \geq 1$.

Step 3: Find a subcompact base:

We now define a collection of $\tau|_X$-open subsets of X by

$$\mathcal{S} := \{B(x_k, j) \cap X : x_k \in X_0, k \leq j < \omega\} \cup \{B(x, n) \cap X : x \in X_1, n \geq 1\}.$$

We see that $\mathcal{S}$ is a base for (X, τ) because the sets $B(x, k) \in \mathcal{S}$ have the right "shape" determined by which of I, R, L or E the point x belongs to, and because the diameter of the sets $B(x, n)$ approaches zero as $n \to \infty$.

What remains is to prove that $\mathcal{S}$ is a subcompact base for (X, τ). To do that, let $\mathcal{F} \subseteq \mathcal{S}$ be a regular filter (with respect to the topology $\tau|_X$) and suppose for contradiction that $\bigcap \mathcal{F} = \emptyset$. Then no member of $\mathcal{F}$ is a singleton. Let $\mathcal{C} := \{B(x, n) : X \cap B(x, n) \in \mathcal{F}\}$. Because $\mathcal{F}$ is a filter base, the collection $\mathcal{C}$ is centered. Because each member of $\mathcal{C}$ is a bounded subset of $(\mathbb{R}, \lambda)$ we know that there is some point $z \in \bigcap\{\mathrm{cl}_\lambda(B(x,n)) : B(x,n) \in \mathcal{C}\}$. However, $z \notin \bigcap \mathcal{F}$ so there must be some $F_0 \in \mathcal{F}$ with $z \notin F_0$. Write $F_0 = B(x_0, k_0) \cap X$ for some $B(x_0, k_0) \in \mathcal{C}$. Because $\bigcap \mathcal{F} = \emptyset$, the collection $\mathcal{F}$ cannot have any minimal element (with respect to inclusion) so that, starting with the set F_0 chosen above, we can find a sequence $F_n \in \mathcal{F}$ such that F_{n+1} is a proper subset of F_n for each $n \geq 0$. Write $F_n = B(x_n, k_n) \cap X$ with $B(x_n, k_n) \in \mathcal{C}$.

Looking back at Claim 1, we conclude from $\bigcap \mathcal{F} = \emptyset$ that at most finitely many members of the sequence F_n could have been constructed in Step 1 using points $x_n \in X_0$. Discarding those finitely many sets, we may renumber and assume that every set $F_n = B(x_n, k_n) \cap X$ was constructed in Step 2, using points $x_n \in X_1$.

Now Claim 4 shows that for any fixed value of K, only a finite number of points x_n have $k_n = K$. Discarding certain finite sub-sequences of the points x_n we obtain a sequence of pairs $(x_{n_1}, k_{n_1}), (x_{n_2}, k_{n_2}), \cdots$ with the property that $k_{n_1} < k_{n_2} < \cdots$. Renumbering the pairs if necessary, we may assume that the sequence of pairs $(x_1, k_1), (x_2, k_2), \cdots$ has $k_1 < k_2 < \cdots$. But then we have $z \in \mathrm{cl}_\lambda(B(x_n, k_n)) \subseteq G(k_n)$. Because the open sets $G(j)$ are nested and have $X_1 = \bigcap\{G(j) : j \geq 1\}$ we obtain $z \in \bigcap\{G(k_n) : n \geq 1\} = X_1 \subseteq X$. Recall that $z \notin F_n = B(x_n, k_n) \cap X$. Because $z \in X$, it follows that $z \notin B(x_n, k_n)$. However, $z \in \mathrm{cl}_\lambda(B(x_n, k_n))$ so that z must be an end point of $B(x_n, k_n)$. Because $z \notin F_n = B(x_n, k_n) \cap X$ while $x_n \in F_n$ we know that $z \neq x_n$. Therefore z is an external endpoint of each set $B(x_n, k_n)$. But that is impossible because, for example, no external endpoint of the set $B(x_1, k_1)$ can be an external endpoint of any set constructed at any later level in the Step 2 recursion. In particular, because $k_1 < k_2$ the sets $B(x_1, k_1)$ and $B(x_2, k_2)$ cannot have any external endpoints in common. This contradiction shows that $\mathcal{S}$ cannot contain any regular (with respect to τ) filter base $\mathcal{F}$ having $\bigcap \mathcal{F} = \emptyset$. Therefore, $\mathcal{S}$ is a subcompact base for (X, τ), as required. $\square$

A well-known theorem shows that a completely metrizable space X is a G_δ-subset of any other metric space that contains X as a subspace. Our next proposition makes an analogous assertion about subcompact GO-spaces constructed on sets of real numbers, namely that a subcompact GO-space X defined on a set of real numbers must be a G_δ-subset of any other GO-space on a set of real numbers that contains X as a subspace. This is, in some sense, a converse of Theorem 3.3. We begin with an easy lemma.

Lemma 3.4 *Let $<$ be the usual ordering of $\mathbb{R}$. If $X \subseteq \mathbb{R}$ and if $(X, <, \sigma)$ is any GO-space, then there is at least one GO-topology τ on $(\mathbb{R}, <)$ with $\sigma = \tau|_X$. In addition, we can choose τ with the properties that*

a) for $x \in X$, $\{x\} \in \sigma$ if and only if $\{x\} \in \tau$ and

b) each point $x \in \mathbb{R} - X$ has a base of neighborhoods of the form (a, b) where $a < x < b$. $\square$

Proposition 3.5 *Suppose σ is a GO-topology on a subset $X \subseteq \mathbb{R}$ and that (X, σ) is subcompact. Let τ be any GO-topology on $\mathbb{R}$ with the property that $\sigma = \tau|_X$. Then X is a G_δ-subset of $(\mathbb{R}, \tau)$.*

Proof: The proof is an application of König's lemma. We begin by noting that because τ is a GO-topology on $\mathbb{R}$, the space $(\mathbb{R}, \tau)$ is hereditarily paracompact because $(\mathbb{R}, \tau)$ has a G_δ-diagonal. Therefore, if $\mathcal{V}$ is any collection of τ-open subsets of $\mathbb{R}$, there is a point-finite collection $\mathcal{W}$ of τ-open sets such that $\bigcup \mathcal{W} = \bigcup \mathcal{V}$, for each $W \in \mathcal{W}$, the τ-closure of W is contained in some member of $\mathcal{V}$ [2], [10].

Let $\mathcal{B}$ be a subcompact base for (X, σ) and let $\mathcal{B}(1) = \{B \in \mathcal{B} : diam(B) < 1\}$, where diameter is computed using the usual metric for $\mathbb{R}$. For each $B \in \mathcal{B}(1)$ there is a set $C(B, 1) \in \tau$ that has diameter < 1 and has $B = C(B, 1) \cap X$. Let $\mathcal{C}(1) = \{C(B, 1) : B \in \mathcal{B}(1)\}$. Then there is a point-finite collection $\mathcal{D}(1) \subseteq \tau$ such that $\bigcup \mathcal{D}(1) = \bigcup \mathcal{C}(1)$ with the property that for each $D \in \mathcal{D}(1)$, some set $\gamma_1(D) \in \mathcal{C}(1)$ has $D \subseteq \mathrm{cl}_\tau(D) \subseteq \gamma_1(D)$.

Suppose $n \geq 1$ and $\mathcal{D}(n)$ is defined. Let

$$\mathcal{B}(n+1) = \{B \in \mathcal{B} : diam(B) < \frac{1}{n+1} \text{ and for some } D \in \mathcal{D}(n), B \subseteq D\}.$$

For each $B \in \mathcal{B}(n+1)$ choose some $\delta(B, n) \in \mathcal{D}(n)$ with $B \subseteq \delta(B, n)$, and then find $C(B, n+1) \in \tau$ with diameter $< \frac{1}{n+1}$ and having

$$B = X \cap C(B, n+1) \subseteq C(B, n+1) \subseteq \delta(B, n).$$

Let $\mathcal{C}(n+1) = \{C(B, n+1) : B \in \mathcal{B}(n+1)\}$. Then there is a point-finite collection $\mathcal{D}(n+1) \subseteq \tau$ with $\bigcup \mathcal{D}(n+1) = \bigcup \mathcal{C}(n+1)$ and

(1) for each $D \in \mathcal{D}(n+1)$ there is some $\gamma_{n+1}(D) \in \mathcal{C}(n+1)$ with $D \subseteq \mathrm{cl}_\tau(D) \subseteq \gamma_{n+1}(D)$.

Let $G_n = \bigcup \mathcal{D}(n)$. Then $G_n \in \tau$ and we have $X \subseteq \bigcap\{G_n : n \geq 1\}$. We claim that $X = \bigcap\{G_n : n \geq 1\}$. To verify that assertion, consider any $q \in \bigcap\{G_n : n \geq 1\}$. For each $n \geq 1$ the collection $\mathcal{D}(n, q) := \{D \in \mathcal{D}(n) : q \in D\}$ is non-empty and finite. Hence so is the collection $\mathcal{K}(n) := \{\gamma_n(D) : D \in \mathcal{D}(n, q)\}$ where $\gamma_n(D)$ is the member of $\mathcal{C}(n)$ chosen in the recursive construction above. Let $\mathcal{K} := \bigcup\{\mathcal{K}(n) : n \geq 1\}$ and define a partial order $\preceq$ on $\mathcal{K}$ by the rule that $K_1 \preceq K_2$ means $\mathrm{cl}_\tau(K_2) \subseteq K_1$. We claim that there is a sequence $K_n = \gamma_n(D_n) \in \mathcal{K}(n)$ with $K_n \preceq K_{n+1}$ for all $n \geq 1$. Once we show that for each $K' \in \mathcal{K}(n+1)$, some $K'' \in \mathcal{K}(n)$ has $K'' \preceq K'$, then the existence of this sequence will follow from a version of König's Lemma (see Theorem 114 in [11] for a result that resembles the more familiar version in Theorem 2.6 of [13]; we need the more general result because we do not claim that $\mathcal{K}$ is a tree).

So fix $K' \in \mathcal{K}(n+1)$, say $K' = \gamma_{n+1}(D')$ where $D' \in \mathcal{D}(n+1, q)$. Then $q \in D'$. Because $\gamma_{n+1}(D') \in \mathcal{C}(n+1)$ we know that there is some $B' \in \mathcal{B}(n+1)$ with $\gamma_{n+1}(D') = C(B', n+1)$. Because $B' \in \mathcal{B}(n+1)$ the set $\delta(B', n) \in \mathcal{D}$ is defined and has $B' \subseteq \delta(B', n)$, and, from the construction of $C(B', n+1)$ we also know that $C(B', n+1) \subseteq \delta(B', n)$. Write $D'' = \delta(B', n)$. Because $D'' \in \mathcal{D}(n)$ we have some $\gamma_n(D'') \in \mathcal{C}(n)$ with $D'' \subseteq \mathrm{cl}_\tau(D'') \subseteq \gamma_n(D'')$. Then we have

$$(2) \quad q \in D' \subseteq \gamma_{n+1}(D') = C(B', n+1) \subseteq \delta(B', n) = D'' \subseteq \mathrm{cl}_\tau(D'') \subseteq \gamma_n(D'').$$

Hence $q \in D'' \in \mathcal{D}(n)$ so that $\gamma_n(D'') \in \mathcal{K}(n)$ and, writing $K'' = \gamma_n(D'')$, we have

$$(3) \quad K' = \gamma_{n+1}(D') \subseteq \mathrm{cl}_\tau(D'') \subseteq \gamma_n(D'') = K''.$$

Therefore, as noted above, König's lemma gives us $K_n = \gamma_n(D_n) \in \mathcal{K}$ with $K_n \preceq K_{n+1}$. i.e., $\gamma_{n+1}(D_{n+1}) \subseteq \mathrm{cl}_\tau(\gamma_{n+1}(D_{n+1})) \subseteq \gamma_n(D_n)$. This gives

(4) $X \cap \gamma_{n+1}(D_{n+1}) \subseteq X \cap \mathrm{cl}_\tau(\gamma_{n+1}(D_{n+1})) \subseteq \gamma_n(D_n)$.

Looking back at the recursive construction above, we see that for each $n \geq 1$, $\gamma_n(D_n) \in \mathcal{C}(n)$ so that for some $B_n \in \mathcal{B}(n)$ we have $\gamma_n(D_n) = C(B_n, n)$. Recall that the set $C(B_n, n)$ was chosen in such a way that $X \cap C(B_n, n) = B_n$ so that assertion (4) gives

(5) $B_{n+1} = X \cap C(B_{n+1}, n+1) \subseteq X \cap \mathrm{cl}_\tau(C(B_{n+1}, n+1)) \subseteq X \cap C(B_n, n) = B_n$.

Therefore $\mathrm{cl}_\sigma(B_{n+1}) \subseteq B_n$ because $\sigma = \tau|_X$, so that the collection $\{B_n : n \geq 1\}$ is a regular filter base in the subcompact base $\mathcal{B}$ for (X, σ). Therefore, some point $r \in X$ has $r \in B_n$ for each $n \geq 1$. But then $r \in B_n \subseteq C(B_n, n) = \gamma_n(D_n)$ and $q \in \gamma_n(D_n) = C(B_n, n)$. Because the diameter of $C(B_n, n)$ is less than $\frac{1}{n}$, it follows that $q = r \in X$, as required to show that $\bigcap\{G_n : n \geq 1\} = X$. $\Box$

One must be careful in applying Proposition 3.5 because, as Example 3.9 will show, there is a dense-in-itself, subcompact GO-space (X, σ) with $X \subseteq \mathbb{R}$ where X is not a G_δ-subset of the usual real line. That example is consistent with Proposition 3.5 because the given σ will not be a relativized topology from $(\mathbb{R}, \lambda)$.

Our next result says that question Q2 has an affirmative answer for GO-spaces constructed on sets of real numbers.

Corollary 3.6 *Suppose $X \subseteq \mathbb{R}$ and suppose that σ is a GO-topology on the linearly ordered set $(X, <)$ such that (X, σ) is subcompact. Suppose that Y is a G_δ-subset of (X, σ). Then $(Y, \sigma|_Y)$ is subcompact.*

Proof: Let τ be any GO-topology on $\mathbb{R}$ that has $\sigma = \tau|_X$. By Proposition 3.5, the set X is a G_δ-subset of $(\mathbb{R}, \tau)$. But then Y is also a G_δ-subset of $(\mathbb{R}, \tau)$ so that by Theorem 3.3, the subspace $(Y, \tau|_Y)$ is subcompact. But because $\tau|_X = \sigma$ we have $\tau|_Y = \sigma|_Y$, as required to show that $(Y, \sigma|_Y)$ is subcompact. $\Box$

Corollary 3.7 *Suppose X is a G_δ-subset of the usual real line $(\mathbb{R}, \lambda)$, and that σ is any GO-topology constructed on the linearly ordered set $(X, <)$. Then (X, σ) is subcompact.*

Proof: As noted in Lemma 3.4, there is at least one GO-topology τ on $\mathbb{R}$ with $\tau|_X = \sigma$. Then X is a G_δ-subset of $(\mathbb{R}, \tau)$ because $\lambda \subseteq \tau$. According to Proposition 3.2, the GO-space $(\mathbb{R}, \tau)$ must be subcompact. According to Theorem 3.3, (X, σ) must be subcompact. $\Box$

Corollary 3.8 *Suppose $X \subseteq \mathbb{R}$. Then (X, σ) is subcompact for every GO-topology σ constructed on $(X, <)$ if and only if X is a G_δ-subset of the usual space of real numbers.*

Proof: Let λ denote the usual topology on $\mathbb{R}$. Then $\lambda|_X$ is a GO-topology on X, and if $(X, \lambda|_X)$ is subcompact, then $(X, \lambda|_X)$ is completely metrizable and therefore is a G_δ-subset of $(\mathbb{R}, \lambda)$. The converse is Corollary 3.7. $\Box$

Corollary 3.7 gives sufficient, but not necessary, conditions for a GO-space (X, σ) on a subset $X \subseteq \mathbb{R}$ to be subcompact, as our next example shows.

Example 3.9 *There is a dense-in-itself subspace X of the Sorgenfrey line $(\mathbb{R}, \sigma)$ that is subcompact even though X is not a G_δ-subset of $(\mathbb{R}, \lambda)$.*

Proof: Let $I_1, I_2, \cdots$ be a listing of the open intervals removed from $[0,1]$ in the usual Cantor set construction. Thus, $I_1 = (\frac{1}{3}, \frac{2}{3}), I_2 = (\frac{1}{9}, \frac{2}{9}), I_3 = (\frac{7}{9}, \frac{8}{9}), \cdots$. Let J_n be the set I_n together with its left endpoint s_n. In the relative Sorgenfrey topology, each J_n is open and subcompact, so that the set $X := \bigcup\{J_n : n \geq 1\}$ is subcompact in the relative Sorgenfrey topology. Note that $\bigcup\{I_n : n \geq 1\}$ is an F_σ-subset of $(\mathbb{R}, \lambda)$ so that if X were a G_δ-subset of $(\mathbb{R}, \lambda)$, then $X - \bigcup\{I_n : n \geq 1\}$ would also be a G_δ-subset of $(\mathbb{R}, \lambda)$. But $X - \bigcup\{I_n : n \geq 1\}$ is the countable, dense-in-itself set $\{s_n : n \geq 1\}$, and no countable dense-in-itself set can be a G_δ-subset of $(\mathbb{R}, \lambda)$. $\square$

Recall Question Q2 of the introduction: suppose the topological space (X, σ) is subcompact and we create a new topology by isolating all of the points in some set $S \subseteq X$ (i.e., we let σ^S be the topology on X having the collection $\sigma \cup \{\{x\} : x \in S\}$ as a base). Must (X, σ^S) be subcompact? If we consider only GO-topologies on subsets or $\mathbb{R}$, our next result provides an affirmative answer.

Proposition 3.10 *Suppose $X \subseteq \mathbb{R}$ and suppose σ is a GO-topology on X such that (X, σ) is subcompact. Let σ^S be obtained from σ by isolating all points in some subset $S \subseteq X$. Then (X, σ^S) is also subcompact.*

Proof: Because (X, σ) is a GO-space, there is a GO-topology τ on $\mathbb{R}$ with $\tau|_X = \sigma$. Because (X, σ) is subcompact, Proposition 3.5 shows that X is a G_δ-subset of $(\mathbb{R}, \tau)$. Let τ^S be the topology having $\tau \cup \{\{x\} : x \in S\}$ as a base. Then τ^S is a GO-topology on $\mathbb{R}$ with $\tau^S|_X = \sigma^S$, and because $\tau \subseteq \tau^S$, X is a G_δ-subset of the GO-space $(\mathbb{R}, \tau^S)$. Now apply Corollary 3.6 to show that (X, σ^S) is subcompact. $\square$

4 Applications to other completeness properties

As noted in Section 2, subcompactness is just one of a cluster of strong completeness properties introduced by de Groot and his Amsterdam colleagues. In general spaces it is easy to see that

(*) regularly co-compact $\Rightarrow$ base compact $\Rightarrow$ subcompact.

The property "co-compact" does not appear in hierarchy (*): the Sorgenfrey line is a co-compact space with respect to the collection $\{[a, b] : a < b\}$ but it is neither regularly co-compact nor base compact [1]. However, the Sorgenfrey line is subcompact, and that is no accident because we can prove:

Proposition 4.1 *Suppose that $X \subseteq \mathbb{R}$ and that τ is a GO-topology constructed on $(X, <)$. If (X, τ) is co-compact, then (X, τ) is subcompact.*

Proof: Let $\mathcal{C}$ be the collection of closed subsets of (X, τ) with respect to which (X, τ) is co-compact. Lemma 3.4 gives a GO-topology σ on $(\mathbb{R}, <)$ such that $\sigma|_X = \tau$, $\{x\} \in \sigma$ if and only if $\{x\} \in \tau$, and has the property that for $x \in \mathbb{R} - X$, basic σ-neighborhoods of x have the form (a, b) for $a < x < b$.

Let $Z = \mathrm{cl}_\sigma(X)$. We will show that Z is a G_δ-subset of $(\mathbb{R}, \sigma)$ and that X is a G_δ-subset of $(Z, \sigma|_X)$ from which it will follow that that X is a G_δ-subset of $(\mathbb{R}, \sigma)$. Then (3.2) and (3.6) combine to show that $(X, \sigma|_X)$ is subcompact, and the proof will be complete because $\sigma|_X = \tau$.

To show that Z is a G_δ-subset of $(\mathbb{R}, \sigma)$, note that if $y \in \mathbb{R} - Z$ then basic σ-neighborhoods of y have the form (a, b) with $a < y < b$. Hence there are rational numbers r, s with $y \in [r, s] \subseteq \mathbb{R} - Z$ so that $\mathbb{R} - Z$ is the union of countably many closed intervals with rational endpoints. Hence $\mathbb{R} - Z$ is an F_σ-subset of $(\mathbb{R}, \sigma)$, so Z is a G_δ, as required.

Next we show that X is a G_δ-subset of $(Z, \sigma|_Z)$. With $\mathcal{C}$ being the collection of closed subsets of (X, τ) given by the definition of co-compactness, there is no loss of generality if we assume that $\mathrm{Int}_\tau(C) \neq \emptyset$ for each $C \in \mathcal{C}$. Let $\mathcal{C}_n := \{C \in \mathcal{C} : \mathit{diam}(C) < \frac{1}{n}\}$ where $\mathit{diam}(C)$ is computed with respect to the usual metric on $\mathbb{R}$. Because $\tau = \sigma|_X$, for each $C \in \mathcal{C}_n$, there is a set $G_n(C) \in \sigma$ with $X \cap G_n(C) = \mathrm{Int}_\tau(C)$, and we may assume that $\mathit{diam}(G_n(C)) < \frac{1}{n}$. Note that $G_n(C) \cap Z \in \sigma|_Z$, so that the set $H_n := \bigcup\{G_n(C) \cap Z : C \in \mathcal{C}_n\} \in \sigma|_Z$ and $X \subseteq H_n$ for each n.

We claim that $\bigcap\{H_n : n \geq 1\} \subseteq X$. Let $y \in \bigcap\{H_n : n \geq 1\}$. Then for each n there is some $C_n \in \mathcal{C}_n$ with $y \in G_n(C_n) \cap Z$. For each $n \geq 1$, $y \in \bigcap\{G_j(C_j) : 1 \leq j \leq n\}$ so that $\bigcap\{G_j(C_j) \cap Z : 1 \leq j \leq n\}$ is a nonempty set in $\sigma|_Z$. We know that X is a dense subset of $(Z, \sigma|_Z)$ so that $\emptyset \neq \bigcap\{G_j(C_j) \cap X : 1 \leq j \leq n\} \subseteq \bigcap\{\mathrm{Int}_\tau(C_j) : 1 \leq j \leq n\} \subseteq \bigcap\{C_j : 1 \leq j \leq n\}$. Therefore, $\{C_n : n \geq 1\}$ is a centered subcollection of $\mathcal{C}$ so that some $x \in X$ has $x \in \bigcap\{C_n : n \geq 1\}$ by co-compactness of (X, τ). Choose $s_n \in \mathrm{Int}_\tau(C_n)$. Then $|x - s_n| < \frac{1}{n}$ because $\mathit{diam}(C_n) < \frac{1}{n}$. Furthermore, $y \in G_n(C_n) \cap Z$ and $s_n \in \mathrm{Int}_\tau(C_n) = G_n(C_n) \cap X \subseteq G_n(C_n) \cap Z$ so that $|y - s_n| \leq \mathit{diam}(G_n(C_n)) < \frac{1}{n}$. Therefore $|y - x| \leq \frac{2}{n}$. Because this holds for each $n \geq 1$ we conclude that $y = x \in X$. Consequently $X = \bigcap\{H_n : n \geq 1\}$ so that X is a G_δ-subset of $(Z, \sigma|_Z)$.

At this point we know that $(X, \tau) = (X, \sigma|_X)$ is a G_δ-subset of $(\mathbb{R}, \sigma)$ so that we may apply Proposition 3.2 and Corollary 3.6 to complete the proof of Proposition 4.1. □

As noted in the Introduction, one of the most interesting open questions in completeness theory is whether every domain-representable space is subcompact. Among general spaces, we expect a negative answer even though in special classes like metrizable, Moore, and BCO spaces, the two notions are equivalent. We do not know whether domain-representability and subcompactness are the same in the category of GO-spaces constructed on subsets of $\mathbb{R}$. However, we can prove:

Proposition 4.2 *Suppose Y is a subset of $\mathbb{R}$ that is domain-representable when equipped with some dense-in-itself GO-topology τ. Then there is a subset $S \subseteq Y$ that is dense in (Y, τ) and is a G_δ-subset of the usual real line.*[4]

Proof: Let X be the closure of the set Y in $(\mathbb{R}, \lambda)$ where λ is the usual open-interval topology on $\mathbb{R}$. In this proof, X always carries the subspace topology $\lambda|_X$ from $\mathbb{R}$. Note that $\lambda|_Y := \{U \cap Y : U \in \lambda\} \subseteq \tau$. Because (Y, τ) has no isolated points, if $G \subseteq Y$ is τ-open, then there is a $\lambda|_X$-open set $H \subseteq X$ with the property that $H \cap Y \subseteq G$ and $H \cap Y$ is a τ-dense subset of G. We will repeatedly use the fact that any base for any space contains a maximal pairwise disjoint subcollection whose union is dense in the space.

Let $(P, \sqsubseteq)$ be a domain that represents (Y, τ). We will abuse notation and write $Y = \max(P)$.

[4]Note that S must also be dense in Y when Y carries the usual subspace topology from $\mathbb{R}$.

Let $P(1) := \{p \in P : 0 < diam(\Uparrow(p) \cap Y) < 1\}$ where diameter is computed by the usual metric on $\mathbb{R}$. Then $\mathcal{B}(1) := \{\Uparrow(p) \cap Y : p \in P(1)\}$ is a base for (Y, τ) so that there is a subset $P'(1) \subseteq P(1)$ with the property that $\mathcal{C}(1) := \{\Uparrow(p) \cap Y : p \in P'(1)\}$ is a maximal pairwise disjoint subcollection of $\mathcal{B}(1)$. Then $\bigcup \mathcal{C}(1)$ is a dense subset of (Y, τ) and therefore also a dense subset of $(Y, \lambda|_Y)$. Because Y is a $\lambda|_X$-dense subset of X, it follows that $\bigcup \mathcal{C}(1)$ is also a dense subset of $(X, \lambda|_X)$ For each $p \in P'(1)$ there is a $\lambda|_X$-open subset $U(1,p) \subseteq X$ with the property that $U(1,p) \cap Y$ is a τ-dense subset of $\Uparrow(p) \cap Y$ and where $diam(U(1,p)) < 1$. Then the collection $\mathcal{U}(1) = \{U(1,p) : p \in P'(1)\}$ is a pairwise disjoint collection (because Y is dense in $(X, \lambda|_X)$) of $\lambda|_X$-open subsets of X. Write $V(1) = \bigcup \mathcal{U}(1)$. Then $V(1)$ is dense in $(X, \lambda|_X)$.

Suppose we have a pairwise disjoint collection $\mathcal{U}(n) = \{U(n,p) : p \in P'(n)\}$ where $P'(n) \subseteq P$ and each $U(n,p)$ is a λ_X-open set of diameter $< \frac{1}{n}$ and where $U(n,p) \cap Y$ is τ-dense in $\Uparrow(p) \cap Y$ for each $p \in P'(n)$. Fix $p \in P'(n)$ and let $P(n+1,p) = \{q \in P : p \ll q,\ 0 < diam(\Uparrow(q) \cap Y) < \frac{1}{n+1}\}$. Then $\mathcal{B}(n+1,p) = \{\Uparrow(q) \cap Y : q \in P(n+1,p)\}$ is a base for the open subspace $\Uparrow(p) \cap Y$ of (Y, τ) so there is a set $P'(n+1,p) \subseteq P(n+1,p)$ with the property that $\mathcal{C}(n+1,p) = \{\Uparrow(q) \cap Y : q \in P'(n+1,p)\}$ is a maximal pairwise disjoint subcollection of $\mathcal{B}(n+1,p)$. Then $\bigcup \mathcal{C}(n+1,p)$ is a τ-dense subset of $U(p) \cap Y$. For each $q \in P'(n+1,p)$ there is a $\lambda|_X$-open subset $U(n+1,p,q)$ of X such that $U(n+1,q,p) \subseteq U(n,p)$ and $U(n+1,p,q) \cap Y$ is τ-dense in $\Uparrow(q) \cap Y$. Let $P'(n+1) := \bigcup\{P'(n+1,p) : p \in P'(n)\}$. Note that the collection $\mathcal{U}(n+1) = \{U(n+1,p,q) : p \in P''(n), q \in P'(n+1,q,p)\}$ is a pairwise-disjoint collection of $\lambda|_X$-open subsets of X that refines $\mathcal{U}(n)$ and has the property that the set $V(n+1) = \bigcup \mathcal{U}(n+1)$ is $\lambda|_X$-dense in X.

Let $S := \bigcap\{V(n) : n \geq 1\}$. Then, because $(X, \lambda|_X)$ is a complete metric space, we know that the set S is non-empty and is $\lambda|_X$-dense in X. We claim that $S \subseteq Y$. For consider any $x \in S$. For each $n \geq 1$ there is a unique $U(n, p_n) \in \mathcal{U}(n)$ with $x \in U(n, p_n)$. Then $p_n \ll p_{n+1}$ in P so that the set $D = \{p_n : n \geq 1\}$ is a directed subset of the domain P. Hence $\sup(D) \in P$ so that because $Y = \max(P)$ there is some $y \in Y$ with $\sup(D) \sqsubseteq y$. But then for each n we have $p_n \ll p_{n+1} \sqsubseteq \sup(D) \sqsubseteq y$ so that $y \in \Uparrow(p_n) \cap Y$. Because the set $U(n, p_n)$ has diameter less than $\frac{1}{n}$ and $U(n,p) \cap Y$ is τ-dense in $\Uparrow(p_n) \cap Y$, we know that $|x - y| \leq \frac{1}{n}$. Because this holds for each $n \geq 1$ we see that $x = y \in Y$, as required. Hence $S \subseteq Y$.

It follows that the set S is $\lambda|_Y$ dense in Y, but even more is true. For each $n \geq 1$ we know that $\bigcup \mathcal{C}(n)$ is τ-dense in Y. By construction, $V(n) \cap Y$ is also τ-dense in Y, and is also τ-open (because $\lambda|_Y \subseteq \tau$). Because (Y, τ) is domain representable, it is a Baire space, so that $\bigcap\{Y \cap V(n) : n \geq 1\}$ must be τ-dense in Y. But $\bigcap\{V(n) \cap Y : n \geq 1\} \subseteq S \subseteq Y$ so we now know that S is dense in (Y, τ) and is a G_δ-subset of $(X, \lambda|_X)$. But X is closed in $(\mathbb{R}, \lambda)$ and therefore any G_δ-subset of $(X, \lambda|_X)$ is also a G_δ-subset of $\mathbb{R}$. $\square$

We note that the "dense-in-itself" hypothesis is necessary in Proposition 4.2: let Y be any subset of $\mathbb{R}$ that contains no dense subset that is a G_δ in the usual real numbers, e.g, a Bernstein set. Let σ be the discrete topology on Y. Then $(Y, <, \sigma)$ is certainly domain-representable.

Corollary 4.3 *None of the following subsets of* $\mathbb{R}$ *can support a dense-in-itself GO-space that is domain representable: a totally non-meager subset of* $\mathbb{R}$*, a Bernstein set, a Q-set, any subset with cardinality less than* 2^ω*.*

Proof: No such subset of $\mathbb{R}$ can contain a dense G_δ-subset of $\mathbb{R}$. $\square$

Corollary 4.4 *Suppose that X is a subset of $\mathbb{R}$ and that τ is a dense-in-itself topology on X so that (X, τ) is domain representable. Then there is a G_δ-subset Y of the usual space $\mathbb{R}$ such that Y is a dense subset of (X, τ) and $(Y, \tau|_Y)$ is subcompact.*

Proof: Use Proposition 4.2 to find a G_δ-subset Y of the usual real line $(\mathbb{R}, \lambda)$ that is a dense subspace of (X, τ). Then Corollary 3.3 shows that any GO-topology on Y must be subcompact. In particular, $(Y, \tau|_Y)$ is subcompact, as required. □

The set S found in Proposition 4.2 has a special significance, as our next result shows.

Proposition 4.5 *Suppose that σ is a GO-topology on some subset $X \subseteq \mathbb{R}$ and suppose that (X, σ) is dense-in-itself. Then the following are equivalent:*

a) (X, σ) is pseudocomplete in the sense of Oxtoby,

b) there is a G_δ-subset S of $(\mathbb{R}, \lambda)$, where λ is the usual topology on $\mathbb{R}$, such that S is a dense subset of (X, σ),

c) the subspace $(S, \sigma|_S)$ is a dense subcompact subspace of (X, σ).

Proof: To show that (a) implies (b), we start with π-bases $\mathcal{P}(n)$ for (X, σ) that witness pseudo-completeness of (X, σ). We may assume that the diameter (with respect to the usual metric on $\mathbb{R}$) of each member of $\mathcal{P}(n)$ is less than $\frac{1}{n}$. Let Z be the closure of X in the space $(\mathbb{R}, \lambda)$.

Let $\mathcal{P}''(1)$ be a maximal pairwise disjoint subcollection of $\mathcal{P}(1)$. Then $\bigcup \mathcal{P}''(1)$ is dense in (X, σ) and because (X, σ) is dense-in-itself, for each $P \in \mathcal{P}''(1)$ there is a set $Q(P, 1) \in \lambda|_Z$ such that $Q(P, 1) \cap X$ is a dense subset of P in the space (X, σ). Because X is dense in $(Z, \lambda|_Z)$ we know that the collection $\mathcal{Q}(1) := \{Q(P, 1) : P \in \mathcal{P}''(1)\}$ is pairwise disjoint.

Suppose $n \geq 1$ and that we already have the pairwise disjoint collection $\mathcal{Q}(n) \subseteq \lambda|_Z$. Let $\mathcal{P}'(n+1)$ be the collection of all $P \in \mathcal{P}(n+1)$ whose closure in (X, σ) is contained in some member of $\mathcal{Q}(n)$. Note that if $P_n \in \mathcal{P}''(n)$ and $P_{n+1} \in \mathcal{P}'(n+1)$ have $P_n \cap P_{n+1} \neq \emptyset$, then $\mathrm{cl}_\sigma(P_{n+1}) \subseteq P_n$. Also, $\mathcal{P}'(n+1)$ is a π-base for (X, σ). Let $\mathcal{P}''(n+1)$ be a maximal pairwise disjoint subcollection of $\mathcal{P}'(n+1)$. Then $\bigcup \mathcal{P}''(n+1)$ is dense in (X, σ) and for each $P \in \mathcal{P}''(n+1)$ there is some $Q(P, n+1) \in \lambda|_Z$ with $Q(P, n+1) \cap X$ being dense in P in the space (X, σ). We may assume that the diameter of $Q(P, n+1)$ is less than $\frac{1}{n+1}$. Then the collection $\mathcal{Q}(n+1)$ is pairwise disjoint because X is dense in $(Z, \lambda|_Z)$.

Let $G_n = \bigcup \mathcal{Q}(n)$. Then $G_n \in \lambda|_Z$ and $G_n \cap X = \bigcup\{Q(P, n) \cap X : P \in \mathcal{P}''(n)\}$ is dense and open in (X, σ). Because (X, σ) is pseudocomplete and therefore a Baire space, we know that $\bigcap\{G_n \cap X : n \geq 1\}$ is dense in (X, σ). We also know that the set $S := \bigcap\{G_n : n \geq 1\}$ is a G_δ-subset of (Z, λ_Z) and therefore also a G_δ-subset of $(\mathbb{R}, \lambda)$.

We claim that $S \subseteq X$. Let $y \in S$ and for each n, choose the unique set $P_n \in \mathcal{P}''(n)$ with $y \in Q(P_n, n)$. Then $Q(P_{n+1}, n+1) \cap Q(P_n, n) \neq \emptyset$ so that $X \cap (Q(P_{n+1}, n+1) \cap Q(P_n, n)) \neq \emptyset$ showing that $P_{n+1} \cap P_n \neq \emptyset$. But then, as noted above, $\mathrm{cl}_\sigma(P_{n+1}) \subseteq P_n$. It now follows from the pseudo-completeness property that there is some $x \in \bigcap\{P_n : n \geq 1\}$. We claim the $x = y$. For fix any $n \geq 1$ and choose some $z_n \in Q(P_n, n) \cap X$. Because $y, z_n \in Q(P_n, n)$ we know that $|y - z_n| < \frac{1}{n}$.

Because $x, z_n \in P_n$ we know that $|z_n - x| < \frac{1}{n}$. Consequently $|x - y| < \frac{2}{n}$ for each n and therefore $y = x \in X$. Therefore, $S \subseteq X$ as claimed.

But then we have $S = S \cap X = \bigcap\{G_n \cap X : n \geq 1\}$ so that S is both dense in (X, σ) and is a G_δ-subset of $(\mathbb{R}, \lambda)$ as required in (b).

To show that (b) implies (c) use Corollary 3.7.

That (c) implies (a) is part of a more general theorem discussed in [1], namely that if S is a dense subset of a regular space (X, σ) such that $(S, \sigma|_S)$ is subcompact with respect to a base $\mathcal{B}$ of relatively open sets, then the collection $\mathcal{C} := \{C \in \sigma : C \cap S \in \mathcal{B}\}$ is a π-base for (X, σ) and if $C_n \in \mathcal{C}$ has $\mathrm{cl}_\sigma(C_{n+1}) \subseteq C_n$ for each $n \geq 1$, then $\bigcap\{C_n : n \geq 1\} \neq \emptyset$. Consequently, defining $\mathcal{P}(n) = \mathcal{C}$ for each n gives that our space (X, σ) is pseudocomplete. □

Corollary 4.6 *Any domain representable GO-space constructed on a subset $X \subseteq \mathbb{R}$ is pseudocomplete.*

Proof: Combine Proposition 4.2 with Proposition 4.5. □

5 Some questions about GO-spaces on sets of real numbers

The most interesting open question about GO-spaces constructed on sets of real numbers is a special case of the more general question Q3 of the Introduction that asks for an example of a domain-representable space that is not subcompact.

Question 5.1 *Suppose (X, τ) is a GO-space constructed on a set $X \subseteq \mathbb{R}$ and suppose (X, τ) is domain representable. Is (X, τ) subcompact? (Compare Corollary 4.4.)*

Results in this paper allow us to understand the role of subcompactness in GO-spaces constructed on subsets of $\mathbb{R}$, but many questions about the other Amsterdam properties remain open.

Question 5.2 *Characterize subsets $X \subseteq \mathbb{R}$ that admit some dense-in-itself GO-topology τ so that (X, τ) has one of the other Amsterdam properties (co-compactness, regular co-compactness, base-compactness). Characterize those subsets $X \subset \mathbb{R}$ so that every GO-topology on X is one of co-compact, regularly co-compact, and base-compact. (See Corollary 3.8 for the subcompact case.)*

We note that results from [1] show that if a subspace X of the Sorgenfrey line is base-compact, then X is nowhere dense in the usual topology of $\mathbb{R}$.

The strong Choquet game $Ch(X)$ on a space X is an infinite topological game that is closely associated with domain representability in the light of K. Martin's theorem that if X is domain representable, then the non-empty player in $Ch(X)$ has a winning strategy that requires knowledge of at most two previous steps in the game (rather than perfect knowledge of the entire history of the game). See [9] for details. We have an analog of 5.2 for this game-theoretic property:

Question 5.3 *Characterize subsets $X \subseteq \mathbb{R}$ that admit some dense-in-itself GO-topology τ so that in $Ch(X, \tau)$ the nonempty player has a winning strategy (respectively, a winning strategy that depends on only the two previous moves, or depends on the previous move only).*

In any game, a strategy that depends only on the opponent's single previous move is called a *stationary winning strategy.* A result in [4] shows that if X is any regular space with a G_δ-diagonal in which the nonempty player has a stationary winning strategy in $Ch(X)$, then X must be domain representable.

We note that the proof of Corollary 3.8 characterizes those subsets $X \subseteq \mathbb{R}$ with the property that for every GO-topology σ on X the space (X, σ) is domain-representable (respectively, has the property that the non-empty player has a winning strategy in $Ch(X, \sigma)$): they are precisely the G_δ-subsets of the usual topology λ on $\mathbb{R}$.

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