Strong completeness properties in topology

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Abstract: In this paper we describe a family of open questions concerning strong completeness properties associated with the Baire Category Theorem. Some of our questions deal with classical completeness topics such as de Groot's subcompactness property and the property now called Choquet completeness, while others ask about more recent topics such as domain-representability and its relation to classical theories.

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1 Introduction

The primary goal of this paper is to present a family of questions about the relationships between the strong completeness properties defined in the next section and about the basic topology of these properties. In this paper, all spaces are at least regular and T_1.

A space X has the Baire Category Property (BCP) if the intersection of countably many dense open subsets of X is dense in X. The classical Baire Category Theorem guarantees that any complete metric space and any locally compact Hausdorff space has BCP, but many other types of spaces also have the BCP, as can be seen from

Theorem 1.1 *Any product space $\Pi\{X_\alpha : \alpha \in A\}$ has the BCP provided each X_α is Čech-complete. In particular the product space has the BCP if each factor is either locally compact or completely metrizable.*

That result is surprising, if only because there is no restriction on the cardinality of the index set A, and because there are examples showing that the BCP itself is badly behaved under the product operation: in fact there is a metrizable BCP space whose square does not have the BCP [29],[14]. In this proposal, we will say that a topological property P is a *(countable) strong completeness property* if any (countable) product of spaces with property P must have the BCP. Consequently, Theorem 1.1 shows that Čech-completeness is a strong completeness property. For many (but not all[1]) strong completeness properties P, we will see that P is even better behaved than required in the definition of a strong completeness property, in the sense that P implies the BCP and the class of spaces with property P is closed under the formation of arbitrary products.

2 Basic definitions

In the 1960s, three strong completeness properties were designed to explain what is really happening in Theorem 1.1. Oxtoby[27] said that a space X is *pseudocomplete* if it has a sequence $\langle \mathcal{P}(n) \rangle$ of π-bases such that $\bigcap\{P(n) : n \geq 1\} \neq \emptyset$ whenever $P(n) \in \mathcal{P}(n)$ and $\emptyset \neq \mathrm{cl}(P(n+1)) \subseteq P(n)$ for each n. De Groot [17] said that a space is *subcompact* if it has a base $\mathcal{B}$ of non-empty open sets such that $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base[2]. Subsequently de Groot and his colleagues in Amsterdam introduced two related strong completeness properties called base-compactness and co-compactness. A space X is *base compact* if it has a base $\mathcal{B}$ with the property that $\bigcap\{\mathrm{cl}(C) : C \in \mathcal{C}\} \neq \emptyset$ whenever $\mathcal{C}$ is a centered[3] subcollection of $\mathcal{B}$, and X is *co-compact* if it has a collection $\mathcal{D}$ of closed sets such that any centered subcollection of $\mathcal{D}$ has non-empty intersection, and such that if U is open in X and $x \in U$, then some $D \in \mathcal{D}$ has $x \in \mathrm{Int}(D) \subseteq D \subseteq U$. If each $D \in \mathcal{D}$ is the closure of its own interior, then we say that X is *regularly cocompact*. These three properties – subcompactness, base-compactness, and co-compactness – have come to be known as *the Amsterdam properties*, and [2] presents a survey. In [13], Choquet introduced a game related to the BCP that is now called *the strong Choquet game on X* and is denoted by $Ch(X)$. Player 1 begins the game by specifying a pair (U_1, x_1) where $x_1 \in U_1$ and U_1 is open in X, and player 2 responds with an open set V_1 that must have $x_1 \in V_1 \subseteq U_1$. Player 1 then chooses a pair (U_2, x_2) with $x_2 \in U_2 \subseteq V_1$ and player 2 responds with an open set V_2 having $x_2 \in V_2 \subseteq U_2$. This game continues, generating a sequence $U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ and player 2 wins the play if $\bigcap\{V_n : n \geq 1\} \neq \emptyset$ (equivalently $\bigcap\{U_n : n \geq 1\} \neq \emptyset$). Because the literature is very confusing about the names for players 1 and 2, we will refer to player 2 as *the non-empty player*. The key question about $Ch(X)$ is whether the non-empty player has a *winning strategy* in the game, i.e., a decision process σ so that if $V_n = \sigma(U_1, x_1, V_1, \cdots, U_n, x_n)$ for each n, then the non-empty player wins, no matter what the other player does. If the non-empty player has a winning strategy, then X is *Choquet complete*. Of particular interest is the case where the strategy σ

[1] Čech-completeness is a strong completeness property that is not arbitrarily productive.

[2] i.e., given $B_1, B_2 \in \mathcal{F}$, then some $B_3 \in \mathcal{F}$ has $\mathrm{cl}(B_3) \subseteq B_1 \cap B_2$

[3] i.e., has the finite intersection property

uses nothing but the pair (U_n, x_n) in choosing the response V_n, and in that case σ is called a *stationary strategy* for the non-empty player. For example, if the space X were locally compact, then for each pair (U_n, x_n) the non-empty player could choose any open set V_n with $x_n \in V_n \subseteq \mathrm{cl}(V_n) \subseteq U_n$, where $\mathrm{cl}(V_n)$ is compact, and that would be a stationary winning strategy.

In the 1970s and 1980s, researchers began to study completeness using sieves of various kinds. The most general type was introduced by E. Michael in [25] and we follow his notation. A *sieve* for a space X is an indexed family of not-necessarily-open covers $\mathcal{S}(n) = \{S(\alpha) : \alpha \in A(n)\}$ together with a sequence of functions $\pi_n : A(n+1) \rightarrow A(n)$ called *bonding maps* with the property that $S_\alpha = X$ if $\alpha \in A(0)$ and if $\alpha \in A(n)$ then $S(\alpha) = \bigcup\{S(\beta) : \beta \in A(n+1), \pi_n(\beta) = \alpha\}$. A sequence of indexes $\langle \alpha_n \rangle$ is a *bonded sequence* if $\pi_n(\alpha_{n+1}) = \alpha_n$ for each n, and the sieve is *complete* if $\bigcap\{\mathrm{cl}(F) : F \in \mathcal{F}\} \neq \emptyset$ whenever $\mathcal{F}$ is a filter base on X and $\langle \alpha_n \rangle$ is a bonded sequence with the property that each $S(\alpha_n)$ contains some member of $\mathcal{F}$. A sieve $\langle \mathcal{S}(n) \rangle$ is *exhaustive (in X)* if for every subset $Y \subset X$ and for every n, Y contains a relatively open subset of the form $Y \cap S$ where $S \in \mathcal{S}(n)$. Clearly an exhaustive sieve is a generalization of a sieve whose members are open covers, something previously studied by Chaber, Choban, and Nagami [12], and by Wicke and Worrell [36]. Michael's goal was to give new insight into the preservation of metric completeness by certain types of mappings, and a few years later Telgarsky and Wicke [34] showed that Michael's completeness, which they renamed *partition completeness*, is preserved by countable products, by perfect mappings and open mappings, by perfect inverse images and is inherited by G_δ-subspaces. Because it is easy to show that any space with a complete exhaustive sieve has the BCP, the results of Telgarsky and Wicke showed that partition completeness is a countable strong completeness property.

Starting in the 1980s, topologists began to explore the role of domain representability in topology. Domain representability is a property designed by D. Scott [31] for use in theoretical computer science. We begin with a poset $(D, \sqsubseteq)$ and say that D is *directed-complete* provided for each non-empty directed subset $E \subseteq D$, some $p \in D$ has $p = \sup(E)$. Using the given relation $\sqsubseteq$ of D we define an auxiliary relation $\ll$ by the rule that $a \ll b$ if for every non-empty directed set E with $b \sqsubseteq \sup(E)$, some $e \in E$ has $a \sqsubseteq e$. With $\Downarrow(b) := \{a \in D : a \ll b\}$ for each $b \in D$, we say that D is *continuous* provided each $\Downarrow(b)$ is directed and has $b = \sup(\Downarrow(b))$. By a *domain* we mean a directed-complete, continuous poset. In any domain, the collection of all sets of the form $\Uparrow(a) := \{b \in D : a \ll b\}$ is a base for a topology on D that is called the *Scott topology.* It is easy to see that any directed complete poset D has a non-empty set of maximal elements (denoted by $\max(D)$). To say that a topological space X is *domain-representable* means that there is a domain D such that X is homeomorphic to $\max(D)$ with the relative Scott topology. It is not hard to prove that any product of domain representable spaces is domain representable and that any domain representable space has the BCP so that, in our terminology, domain representability is a strong completeness property. In fact, K. Martin [23] proved more: if X is domain representable, then the non-empty player has a winning strategy in the strong Choquet game $Ch(X)$, so that X is Choquet complete.

3 Subcompactness and domain representable spaces

It is surprising that, after almost 50 years, open questions still remain about de Groot's subcompactness and the other Amsterdam properties. The most fundamental is

Question 3.1 *Suppose X is subcompact and Y is a (dense) G_δ-subset of X. Is Y subcompact? In particular, must every Čech-complete space be subcompact?*[4]

A second open question requires some special notation. Suppose τ is a topology on a space X and $S \subseteq X$. Let τ^S be the topology on X whose base is $\tau \cup \{\{x\} : x \in S\}$. This new topology isolates all points of S and agrees with τ on $X - S$, and we have

Question 3.2 *Suppose that (X, τ) is subcompact and $S \subseteq X$. Is (X, τ^S) subcompact?*

As it happens, these two questions are closely related to a newer question. In [3] we proved:

Proposition 3.3 *If X is a subcompact regular space, then X is domain representable.*

We do not know whether the converse of Proposition 3.3 is true and we have:

Question 3.4 *Is it true that every domain representable space is subcompact?*

We expect a negative answer to Question 3.4 among general spaces, but as explained below, we have shown that subcompactness and domain representability are equivalent in many types of spaces with additional structure. In the light of our next result (proved in [4]), a negative answer to either of Questions 3.1 or 3.2 would give a negative answer to Question 3.4 because we proved:

Proposition 3.5 *Suppose (X, τ) is domain representable. Then any G_δ-subset of X is also domain representable and for any $S \subseteq X$, the space (X, τ^S) is domain representable.*

Proposition 3.5 suggests that it is easier to answer questions about domain representable spaces than about subcompact spaces but there is at least one exception. It is known [2] that a locally subcompact space must be subcompact, and the following question remains open:

Question 3.6 *If $X = U \cup V$ where U and V are open, domain representable subspaces of X, is X domain representable? More generally, if X is locally domain representable, is X domain representable?*

As mentioned above, Question 3.4 is known to have an affirmative answer for certain classes of spaces. In [10] we proved:

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[4] In a private communication in 2008, Fleissner showed that if $\underline{c} = 2^\omega$ and if D is a countable subset of $X = \{0,1\}^{\underline{c}}$, then $X - D$ is subcompact.

Theorem 3.7 *Suppose X is a Moore space. Then the following are equivalent:*

a) X has a development $\mathcal{G}(n)$ such that if $G(n) \in \mathcal{G}(n)$ and $\mathrm{cl}(G(n+1)) \subseteq G(n)$, then $\bigcap\{G(n) : n \geq 1\} \neq \emptyset$, i.e., X is Rudin complete*;*

b) X is subcompact;

c) the non-empty player has a winning strategy in the strong Choquet game $Ch(X)$;

d) the non-empty player has a stationary winning strategy in $Ch(X)$;

e) X is domain representable.

It is easy to see that if (X, τ) is a Moore space and $S \subseteq X$, then (X, τ^S) is a Moore space if and only if S is an F_σ-subset of (X, τ). Therefore, the combination of Theorem 3.7 and Theorem 3.5 gives affirmative answers to Questions 3.1, 3.2, and 3.6 in the category of Moore spaces.

Corollary 3.8 *Suppose (X, τ) is a Moore space.*

a) If (X, τ) is subcompact and if Y is a G_δ-subspace of X, then as a subspace of X, Y is also subcompact;

b) If (X, τ) is subcompact, $S \subseteq X$ and (X, τ^S) is also a Moore space, then (X, τ^S) is subcompact.

c) If X is locally domain representable, then X is domain representable.

In fact, the equivalence of (b), (c), (d) and (e) given in Theorem 3.7 actually holds in the wider class of spaces that have a Base of Countable Order (BCO) in the sense of Wicke and Worrell [35].

In [3] we showed that every completely quasi-developable space[5] is domain representable. The Michael line shows that a subcompact quasi-developable space can fail to be completely quasi-developable. It is easy to see that any Čech complete quasi-developable space is completely quasi-developable, but we do not know whether the converse holds. In addition, the following question remains open. It is a special case of Question 3.4, above.

Question 3.9 *Suppose X is quasi-developable. If X is domain representable, is X subcompact?*

The classes of σ-spaces and semi-stratifiable spaces [18] are also natural generalizations of the class of Moore spaces. While we have already investigated many of the standard examples of such spaces, the following question remains open.

[5]A space X is *completely quasi-developable* if there is a sequence $\mathcal{G}(n)$ of open collections in X that is a quasi-development for X and has the additional property that $\bigcap\{M_k : k \geq 1\} \neq \emptyset$ whenever M_k is a decreasing sequence of nonempty closed sets with the property that for some choice of $n_1 < n_2 < \cdots$, some $G_k \in \mathcal{G}(n_k)$ has $M_k \subseteq G_k$. This property is obviously a relative of Moore completeness in Moore spaces.

Question 3.10 *What are the relations between subcompactness, domain representability, and the strong Choquet game in the class of semi-stratifiable spaces?*

Because any semi-stratifiable space has a G_δ-diagonal, the previous question is linked to questions about the strong Choquet game in a later section.

Unlike the situation in metric spaces, there are at least two types of completeness in Moore spaces. Rudin completeness is defined in part (a) of Theorem 3.7 and *Moore completeness* is defined as follows: there is a development $\langle \mathcal{G}(n) \rangle$ for X such that $\bigcap\{M_n : n \geq 1\} \neq \emptyset$ whenever $\langle M_n \rangle$ is a decreasing sequence of nonempty closed sets such that for each n, M_n is a subset of some member of $\mathcal{G}(n)$. Among completely regular Moore spaces, Moore completeness is equivalent to Čech completeness and is strictly stronger than Rudin completeness[6].

As it happens, there is a special kind of domain called a *Scott domain*[7] that can be used to represent certain spaces, and K. Martin [22] proved that any Moore space X that has $X = \max(D)$ for a Scott domain D must be Čech complete. For some time it was an open question whether Scott-domain representability characterized Moore completeness (in the class of Moore spaces). Then several researchers noted a link between Scott-domain representability and the co-compactness property of de Groot and his Amsterdam colleagues (defined in Section 2). The next result appears in [19] and an easier proof appears in [5].

Proposition 3.11 *Suppose X is homeomorphic to* $\max(D)$ *where* $(D, \sqsubseteq)$ *is a Scott domain. Then X is co-compact with respect to the collection* $\mathcal{D} := \{\uparrow(d) : d \in D\}$.

Given Proposition 3.11, a Moore space constructed by Tall [32] becomes relevant to our study: it is a Čech-complete Moore space that is not co-compact, showing that Moore completeness is not equivalent to Scott-domain representability for Moore spaces. Mishkin [26] showed how to embed Tall's space in a co-compact Moore space as a closed (and therefore G_δ) subset, and in [5] we were able to construct a Scott domain that represents one version of Mishkin's space. This leaves open the following questions:

Question 3.12 *Is it true that every co-compact Moore space is Scott domain representable? Is it true that every Čech-complete Moore space can be embedded (as a closed subset, or as a dense subset) in a co-compact Moore space?*

It is not surprising that some results about domain representability of Moore spaces are axiom-sensitive. For example, constructions in [10] show:

Proposition 3.13 *The statement that each countably paracompact separable Čech-complete Moore space is Scott-domain representable is independent of and consistent with ZFC.*

Question 3.14 *In ZFC, must every normal, separable, Čech-complete Moore space be Scott-domain representable?*

[6] Of course, among metric spaces, the two completeness concepts are equivalent.

[7] A *Scott domain* is a domain D with the extra property that if $d_1, d_2 \in D$ have $d_1, d_2 \sqsubseteq d_3$ for some $d_3 \in D$, then $\sup\{d_1, d_2\}$ exists in D.

What may be important for future work is that Kopperman, Kunzi, and Waszkiewicz [19] combined Proposition 3.11 with a certain bi-topological property called *pairwise complete regularity* to characterize spaces that are Scott domain representable. However, it is not clear how to use this bi-topological condition in Moore spaces.

There are spaces such as $[0, \omega_1)$ that are hereditarily domain representable and we ask:

Question 3.15 *Suppose X is hereditarily domain representable (hereditarily subcompact). Is X scattered?*

The work of Michael [25], and of Telgarsky and Wicke [34] allows us to understand which kinds of mappings preserve partition completeness, but the situation for domain representability is largely unexplored.

Question 3.16 *Which kinds of mappings preserve (Scott) domain representability?*

4 Measurements and domains

If D is a domain, then for each $x \in \max(D)$ the elements of $\Downarrow(x)$ approximate the point x in an order-theoretic and topological sense. K. Martin [22] introduced a numerical measure of the degree of approximation that he called a measurement on the domain. Let $[0, \infty)^*$ denote the set $[0, \infty)$ with the reverse order. Then $[0, \infty)^*$ is a Scott domain with 0 as its unique maximal element, and $[0, \infty)^*$ has a Scott topology that is not the same as the Euclidean topology. By a *measurement* on a domain D we mean a function $\mu : D \to [0, \infty)^*$ that has:

a) μ is continuous when both domain and range carry their Scott topology;

b) if $x \in D$ with $\mu(x) = 0$ and if $\langle p_n \rangle$ is a sequence of elements of $\Downarrow(x)$ having $\lim\{\mu(p_n) : n \to \infty\} = 0$, then $\{p_n : n \geq 1\}$ is a directed set whose supremum is x.

The *kernel* of μ is the set $\ker(\mu) := \{p \in D : \mu(p) = 0\}$ and it is easy to show that $\ker(\mu) \subseteq \max(D)$ and that $\ker(\mu)$ is a G_δ-subset of D in the Scott topology [22].

Question 4.1 *Characterize X so that for some (Scott) domain D, $X = \max(D)$ and $\max(D)$ is a G_δ-subset of D. Characterize those spaces X for which there is a domain (Scott domain) D and a measurement μ on D such that X is homeomorphic to $\ker(\mu)$.*

In [6] we extended results from [22], obtaining some necessary conditions that suggest directions for attacking the first part of Question 4.1:

Theorem 4.2 *Suppose that D is a domain and that $X \subseteq \max(D)$ is a G_δ-subset of D. Then in the relative Scott topology, X is first-countable and domain representable, and is a union of a family of dense G_δ-subspaces, each of which is completely metrizable.*

Theorem 4.3 *Suppose that S is a Scott domain and that X is a G_δ-subset of* $\max(S)$*. Then X is weakly developable in the sense of [1] and therefore has a G_δ-diagonal and a base of countable order in the sense of [35]. In addition, there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers such that if $\mathcal{F}$ is a centered collection of nonempty closed subsets of X and for each n, some $G_n \in \mathcal{G}(n)$ contains a member of $\mathcal{F}$, then $\bigcap \mathcal{F} \neq \emptyset$. Hence, if X is completely regular, then X is Čech-complete. In addition, if X is regular and submetacompact, then X is a complete Moore space, and if X is paracompact or countably compact, then X is metrizable.*

Example 4.4 *Let $X := [0, \omega_1)$ with the usual open interval topology. Then there is a domain D such that $X = \max(D)$ is a G_δ-subset of D, but there is no Scott domain S such that $X = \max(S)$ is a G_δ-subset of S. (This example appears in [6] and corrects an error in the literature [30].)*

Example 4.5 *Burke [11] constructed a locally compact Hausdorff space X with a G_δ-diagonal that is not a Moore space, and there is a Scott domain S such that $X = \max(S)$ is a G_δ-subset of S, and yet X is not the kernel of any measurement on any Scott domain. (This example answers a question posed in [24].)*

Question 4.6 *Suppose X is completely metrizable. Is there a Scott domain S_X such that $X = \max(S_X)$ is a G_δ-subset of S_X? (A potentially easier question from [22] asks whether there is a Scott domain S where X is a dense subset of* $\max(S)$ *and also a G_δ-subset of S. Martin [22] provided an affirmative answer in case X is a complete separable metric space.)*

Question 4.7 *Suppose Y is a Scott-domain-representable Moore space. Is there a Scott domain S_Y such that $Y = \max(S_Y)$ is a G_δ-subset of S_Y? Is every Scott-domain-representable Moore space the kernel of a measurement on some Scott domain?*

5 The strong Choquet game

The strong Choquet game $Ch(X)$ on a space X was described in Section 2. K. Martin [23] was the first to see the relation between domain representability and the strong Choquet game when he proved:

Theorem 5.1 *If X is domain representable, then the non-empty player has a winning strategy in the strong Choquet game $Ch(X)$. In fact, the non-empty player has a winning strategy that depends on knowing only the previous two moves in the game (as opposed to knowing the entire history of the game).*

The claim that the non-empty player needs to know only the previous two moves of $Ch(X)$ raises some delicate issues. Martin begins by selecting a domain P with $X = \max(P)$ and then choosing certain points $p_n \in P$ and then letting the set $V_n := \Uparrow(p_n) \cap \max(P)$ be the non-empty player's response to the pair (U_n, x_n) where $x_n \in U_n$ and U_n is an open set in X. The goal is to get $E := \{p_n : n \geq 1\}$ to be a directed set, so that $\sup(E) \in P$ will lie below (or equal) an element of $\bigcap\{V_n : n \geq 1\}$. Martin points out that to finish the proof, the non-empty player needs to know only the point p_{n-1} that was used to define V_{n-1}, and then to choose p_n so that $p_{n-1} \ll p_n$, and that is the sense in which it is enough for the non-empty player to know only two previous moves in the game[8]. But what if the non-empty player knows only the set V_{n-1} and the pair (x_n, U_n) with $x_n \in U_n \subseteq V_{n-1} \subseteq \max(P)$? It is not clear how the non-empty player can reconstruct the point p_{n-1} used to define V_{n-1} from the potentially many points p with $\Uparrow(p) \cap \max(P) = V_{n-1}$. On the other hand, if (in a domain representable space) the non-empty player can look all the way back to the first move in the game, i.e., to the pair (U_1, x_1), then a well-ordering argument allows the non-empty player to reconstruct the point p_{n-1} that was used to determine V_{n-1}, and then to arrange a point $p_n \in P$ with $p_{n-1} \ll p_n$. Thus we have

Question 5.2 *(a) Suppose that X is domain representable. Does the non-empty player have a winning strategy in $Ch(X)$ that chooses the response V_n based on knowing only the subset $V_{n-1} \subseteq X$ and the pair (U_n, x_n), but not the element of the representing domain P that was used to define V_{n-1}?*

(b) Find an example of a domain representable space in which the non-empty player has no winning stationary strategy in $Ch(X)$.

Debs [16] has constructed a family of topological spaces in which the non-empty player has a winning strategy, but no stationary winning strategy, in the Banach-Mazur game[9]. We now know that one of Debs' spaces is not domain-representable. However, there may be other variations of Debs' space that are domain-representable and if such spaces exist, they would provide the counterexample sought in (b) of Question 5.2.

As noted in the previous section, spaces having a G_δ-diagonal are one of the broadest types of generalized metric spaces, and in [3] we proved:

Proposition 5.3 *Suppose X has a G_δ-diagonal and that the non-empty player has a stationary winning strategy in the strong Choquet game $Ch(X)$. Then X is domain representable.*

Question 5.4 *Suppose X has a G_δ-diagonal.*

a) If the non-empty player has a winning strategy in $Ch(X)$ that might not be a stationary strategy, must X be domain representable? Must X be subcompact?

b) Suppose the non-empty player has a stationary winning strategy in $Ch(X)$. Must X be subcompact?

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[8]In a Scott domain the non-empty player needs to know only the single previous move, namely the pair (U_n, x_n). Any choice of p_n with $x_n \in \Uparrow(p_n) \cap \max(P) \subseteq U_n$ can be used because $\{p_k : 1 \leq k \leq n\}$ is bounded in P by x_n so that $q_n := \sup\{p_k : 1 \leq k \leq n\}$ is in P and can be used to win the game. This shows that in a Scott-domain representable space X, the non-empty player has a stationary winning strategy in $Ch(X)$.

[9]In the Banach-Mazur game, players 1 and 2 alternate choosing terms of a nested sequence of nonempty open sets $U_1, V_1, U_2, V_2, \cdots$ and the nonempty player (who chooses the sets V_n) wins if $\bigcap\{V_n : n \geq 1\} \neq \emptyset$.

Recall that a space X is *Choquet-complete* if the non-empty player has a winning strategy in the strong Choquet game $Ch(X)$, where the strategy is allowed to use the entire previous history of the play. The basic topology of Choquet-complete spaces is well-understood. For example, any G_δ-subspace of such a space is Choquet complete and if (X, τ) is Choquet complete, then so is (X, τ^S) for every subset $S \subseteq X$ (where τ^S is as defined in Section 3). But some basic topological questions remain open about spaces in which the non-empty player has a *stationary* winning strategy in $Ch(X)$. Porada [28] outlined a proof that if X is Čech-complete, then the non-empty player has a stationary winning strategy in $Ch(X)$. His proof is not correct, but it can be fixed, and a further modification gives the first part of our next result:

Proposition 5.5 *Suppose that the non-empty player has a stationary winning strategy in the strong Choquet game $Ch(X, \tau)$ in the space (X, τ).*

a) If Y is a dense G_δ-subspace of X then the non-empty player also has a stationary winning strategy in $Ch(Y)$.

b) If $S \subseteq X$, then the non-empty player has a stationary winning strategy in $Ch(X, \tau^S)$.

Proof: For part (a), write $Y = \bigcap\{G(n) : n \geq 1\}$ where $G(1) = X$ and $G(n+1) \subseteq G(n)$. Define $G(\infty) = Y$. We will use the notational convention $\infty + 1 = \infty$ and we will write $\tau|_Y$ for the subspace topology inherited by Y.

For $a \in H \in \tau$, let $r(H, a) \in \tau$ have $a \in r(H, a) \subseteq \mathrm{cl}_X(r(H, a)) \subseteq H$. For $U \in \tau|_Y$ let $\hat{U} = \bigcup\{H \in \tau : H \cap Y \subseteq U\}$. Then $\hat{U} \cap Y = U$ and U is dense in $\hat{U}$. Also define

$$N(U) = \sup\{k : \hat{U} \subseteq G(k)\}.$$

Then $1 \leq N(U) \leq \infty$ and $\hat{U} \subseteq G(N(U))$.

Let σ be a stationary wining strategy for the non-empty player in the game $Ch(X, \tau)$. Let $y \in U \in \tau|_Y$; then (U, y) is a conceivable move by the empty player in $Ch(Y, \tau|_Y)$. Our goal is to define a set $\psi(U, y) \in \tau|_Y$ with $y \in \psi(U, y) \subseteq U$ in such a way that ψ becomes a stationary winning strategy for the non-empty player in $Ch(Y, \tau|_Y)$. With $N(U)$ as above, consider the pair $(\hat{U} \cap G(N(U)+1), y)$. Using the "regularity operator" r defined above, find the set $r(\hat{U} \cap G(N(U)+1), y) \in \tau$ and then use the stationary winning strategy σ to find $\sigma(r(\hat{U} \cap G(N(U)+1), y), y) \in \tau$. Finally let

$$\psi(U, y) = Y \cap r(\sigma(r(\hat{U} \cap G(N(U)+1), y), y), y).$$

Then $y \in \psi(U, y) \subseteq U$ and $\psi(U, y) \in \tau|_Y$.

<u>Claim 1</u>: Suppose $U, V \in \tau|_Y$ with $y \in U$, $z \in V$ and suppose $V \subseteq \psi(U, y)$. Then $\hat{V} \subseteq G(N(U)+1)$ so that $N(V) \geq N(U)+1$. (Recall the convention that $\infty + 1 = \infty$.) To verify Claim 1 we will show that

$$(*) \quad \hat{V} \subseteq \mathrm{cl}_X(r(\hat{U} \cap G(N(U)+1), y).$$

If (*) fails, then the set $H := \hat{V} - \mathrm{cl}_X(r(\hat{U} \cap G(N(U)+1), y))$ is non-empty and open in X. Because Y is dense in X, there must exist some $y_0 \in H \cap Y$. Because $H \subseteq \hat{V}$ we have $y_0 \in Y \cap H \subseteq Y \cap \hat{V} = V$ and

$$V \subseteq \psi(U, y) \subseteq r(\sigma(r(\hat{U} \cap G(N(U)+1), y), y), y) \subseteq \sigma(r(\hat{U} \cap G(N(U)+1), y), y).$$

But

$$\sigma(r(\hat{U} \cap G(N(U)+1), y), y) \subseteq r(\hat{U} \cap G(N(U)+1), y)$$

showing that $y_0 \in r(\hat{U} \cap G(N(U)+1), y)$ which contradicts $y_0 \in H = \hat{V} - \mathrm{cl}_X(r(\hat{U} \cap G(N(U)+1), y))$. Therefore Claim 1 holds.

Claim 2: Suppose $U, V \in \tau|_Y$ with $y \in U$, $z \in V$ and suppose $V \subseteq \psi(U, y)$. Then in X we have $\hat{V} \subseteq \sigma(r(\hat{U} \cap G(N(U)+1), y), y)$.

To verify Claim 2 we note that $\psi(U, y) \supseteq V$ gives $r(\sigma(r(\hat{U} \cap G(N(U)+1), y), y), y) \supseteq V$ so that $\mathrm{cl}_X(r(\sigma(r(\hat{U} \cap G(N(U)+1), y), y), y) \supseteq \mathrm{cl}_X(V)$. Because Y is dense in X we know that $\mathrm{cl}_X(V) \supseteq \hat{V}$ which gives

$$\hat{V} \subseteq \mathrm{cl}_X(V) \subseteq \mathrm{cl}_X(r(\sigma(r(\hat{U} \cap G(N(U)+1), y), y), y)) \subseteq \sigma(r(\hat{U} \cap G(N(U)+1), y), y),$$

as claimed.

Given Claims 1 and 2, we are ready to show that ψ is a stationary winning strategy in $Ch(Y, \tau|_Y)$. Suppose that the non-empty player uses ψ as a stationary strategy in the game $Ch(Y, \tau|_Y)$ and that

$$(U_1, y_1),\ \psi(U_1, y_1),\ (U_2, y_2),\ \psi(U_2, y_2),\ (U_3, y_3),\ \cdots$$

is a play of the game. By Claim 1, we have $N(U_k) \geq k$ for each k. Now consider the pair $(r(\hat{U}_1 \cap G(N(U_1)+1), y_1), y_1)$. We have $\hat{U}_2 \subseteq \sigma(r(U_1 \cap G(N(U_1)+1), y_1), y_1)$ from Claim 2. More generally, consider the pair $(r(\hat{U}_k \cap G(N(U_k)+1), y_k), y_k)$. Then $\sigma(r(\hat{U}_k \cap G(N(U_k)+1), y_k), y_k) \in \tau$ and Claim 2 gives $\hat{U}_{k+1} \subseteq \sigma(r(\hat{U}_k \cap G(N(U_k)+1), y_k), y_k)$ so that

$$y_{k+1} \in \sigma(r(\hat{U}_k \cap G(N(U_k)+1), y_k), y_k)$$

and therefore

$$\sigma(r(\hat{U}_{k+1} \cap G(N(U_{k+1}+1), y_{k+1}), y_{k+1}) \subseteq \hat{U}_{k+1} \subseteq \sigma(r(\hat{U}_k \cap G(N(U_k)+1), y_k), y_k).$$

To simplify notation, write $H_k := \sigma(r(\hat{U}_k \cap G(N(U_k)+1), y_k), y_k)$. Then we have $y_k \in H_k \in \tau$ and $H_{k+1} \subseteq \hat{U}_{k+1} \subseteq \sigma(H_k, y_k)$. Therefore the sequence

$$(H_1, y_1),\ \sigma(H_1, y_1),\ (H_2, y_2),\ \sigma(H_2, y_2),\ (H_3, y_3)\ \cdots$$

is a play of the game $Ch(X, \tau)$. Because σ is a stationary winning strategy in $Ch(X, \tau)$, the nonempty player wins and we know that $\bigcap\{H_k : k \geq 1\} \neq \emptyset$. By Claim 1, $H_k \subseteq G(N(U_k)) \subseteq G(k)$ so that we have $\bigcap\{H_k : 1 \leq k < \infty\} \subseteq \bigcap\{G(k) : 1 \leq k < \infty\} \subseteq Y$.

Let $z \in \bigcap\{H_k : k \geq 1\}$. Then $z \in H_k \cap Y \subseteq \hat{U}_k \cap Y \subseteq U_k$ for each k, showing that $\bigcap\{U_k : k \geq 1\} \neq \emptyset$, as required to prove part (a) of this theorem.

The proof of part (b) of Proposition 5.5 is easy and is reminiscent of Question 3.2 and Theorem 3.5. □

In the proof of part (a) of Proposition 5.5, it is crucial that Y is dense in X and that raises:

Question 5.6 *Suppose that the non-empty player has a stationary winning strategy in $Ch(X)$ and that Y is a G_δ-subset of X. Without assuming that Y is dense in X, does the non-empty player have a stationary winning strategy in $Ch(Y)$?*

Part of the difficulty in Question 5.6 is that (as the Michael line shows) closed sets do not necessarily inherit Choquet completeness so that one may not assume that the non-empty player has a stationary winning strategy in $Ch(\mathrm{cl}(Y))$. Telgarsky [33] approached Question 5.6 by using a relativized version of $Ch(X)$ that he denoted by $G(X,Y)$ where open sets in X were used to play the strong Choquet game in the subspace Y, and he proved that if the non-empty player has a stationary winning strategy in $Ch(X)$, then the non-empty player also has a winning strategy in $G(X,Y)$ for every non-empty G_δ-subspace Y of X. Telgarsky included a comment that this would give a stationary winning strategy for the non-empty player in $Ch(Y)$, but that assertion was not proved and we have:

Question 5.7 *Suppose the non-empty player has a stationary winning strategy in $Ch(X)$ and in Telgarsky's game $G(X,Y)$ from [33]. Does the non-empty player have a stationary winning strategy in $Ch(Y)$? What if Y is assumed to be a G_δ-subset of X?*

6 Strong completeness in function spaces

In this section, all spaces are at least completely regular. Let $C_p(X)$ be the set of continuous real-valued functions on a space X under the pointwise-convergence topology. Recent results have given a basic understanding of subcompactness, domain representability, and various types of Choquet completeness in $C_p(X)$, but many hard questions remain.

Results of [20] show that is it difficult, but possible, to have $C_p(X)$ satisfy the BCP (see Section 1) for a non-discrete space X. A folklore question asked whether $C_p(X)$ could be subcompact if X is not discrete, and [21] provided a negative answer. In [8] we generalized that result for normal spaces, proving

Theorem 6.1 *Suppose X is a T_4-space. Then the following are equivalent:*

a) $C_p(X)$ is domain representable;

b) $C_p(X)$ is Scott-domain representable;

c) $C_p(X)$ *is subcompact;*

d) X *is discrete.*

Question 6.2 *Can Theorem 6.1 be proved if X is completely regular but not T_4?*

By placing restrictions on the limit point structure of X, it is possible to weaken the normality assumption in Theorem 6.1 to complete regularity plus pseudo-normality[10]. Recall that space X is *pseudo-radial* if for each $Y \subseteq X$, Y fails to be closed if and only if there is a transfinite sequence σ (i.e., a net of points of Y with a well-ordered domain) that converges to some point of $X - Y$. In [7] we proved:

Proposition 6.3 *Suppose X is completely regular and pseudo-normal. If X is pseudo-radial, then the following are equivalent:*

a) $C_p(X)$ *is domain representable;*

b) $C_p(X)$ *is Scott-domain representable;*

c) X *is discrete.*

It would be desirable to generalize Proposition 6.3 because the class of pseudo-radial spaces is quite restrictive. However, it does include the class of GO-spaces that will be mentioned in the next section.

Another way to improve Theorem 6.1 is to look only at Scott-domain representability, and in that case we proved in [8]

Proposition 6.4 *Suppose X is completely regular and pseudo-normal. Then $C_p(X)$ is Scott-domain representable if and only if X is discrete.*

The role of the strong Choquet game in $C_p(X)$ is relatively well understood. In [7] we characterized pseudo-normal spaces for which $C_p(X)$ is Choquet complete by proving:

Proposition 6.5 *Suppose X is completely regular and pseudo-normal. Then the following are equivalent:*

a) *every countable subset of X is closed;*

b) *the non-empty player has a winning strategy in the strong Choquet game in $C_p(X)$;*

c) *the non-empty player has a stationary winning strategy in the strong Choquet game in $C_p(X)$;*

d) $C_p(X)$ *is pseudocomplete in the sense of Oxtoby [27].*

---

[10] A space X is *pseudo-normal* if disjoint closed sets, one of which is countable, have disjoint neighborhoods.

Question 6.6 *Can Proposition 6.5 be proved without assuming that X is pseudo-normal?*

The authors of [21] included questions about spaces for which $C_p(X)$ contains a dense subcompact subspace, and analogous questions exist for domain representable spaces. For example:

Question 6.7 *Must a completely regular space X be discrete if $C_p(X)$ contains a dense subcompact subspace? Must a normal space be discrete if $C_p(X)$ contains a dense domain representable subspace (respectively, a dense Scott-domain representable subspace)? In the second question, what if X is not assumed to be normal?*

7 Strong completeness in GO-spaces

Except where otherwise noted, the results in this section will appear in [9]. A *generalized ordered space* (GO-space) is a triple $(X, <, \tau)$ where $<$ is a linear ordering of a set X and τ is a topology on X having a base of order-convex sets (possibly including singleton sets). GO spaces seem simple, but they have provided many of the central counterexamples in general and set-theoretic topology, e.g., ordinal spaces and the lines named after Souslin, Aronszajn, Sorgenfrey, and Michael, and their variations.

Even when the GO-spaces in question are constructed on the set $\mathbb{R}$ of all real numbers, or on subsets $X \subseteq \mathbb{R}$, the resulting GO-spaces can have interesting completeness properties related to the questions in Section 3 of this proposal. For example, we have

Proposition 7.1 *Any GO-space constructed on the entire set $\mathbb{R}$ is Scott-domain representable. [15]*

Proposition 7.2 *Suppose that τ is any GO topology on the usual ordered set $\mathbb{R}$ of real numbers. Then $(\mathbb{R}, \tau)$ is subcompact. [9]*

The proof of the next proposition given in [9] is quite complicated, and it would be good to have a more direct argument.

Proposition 7.3 *Suppose that τ is a GO-topology on $\mathbb{R}$ and that X is a G_δ-subspace of $(\mathbb{R}, \tau)$. Then $(X, \tau|_X)$ is subcompact.*

Proposition 7.3 allows us to answer Questions 3.1 and 3.2 for GO-spaces constructed on sets of real numbers. We outline the proofs of the next three corollaries; full details appear in [9].

Corollary 7.4 *Suppose σ is any GO-topology on any subset $X \subseteq \mathbb{R}$ and that (X, σ) is subcompact.*

a) If Y is a G_δ-subset in (X, σ), then $(Y, \sigma|_Y)$ is also subcompact.

b) For any subset $S \subseteq X$, the space (X, τ^S) is subcompact.

Proof: To prove assertion (a), first observe that there is a GO-topology $\hat{\sigma}$ on $\mathbb{R}$ with the properties that

i) $\hat{\sigma}|_X = \sigma$

ii) if $x \in X$ then $\{x\} \in \hat{\sigma}$ if and only if $\{x\} \in \sigma$

iii) each $y \in \mathbb{R} - X$ has a neighborhood base in $\hat{\sigma}$ consisting of sets of the form (a, b) where $a < y < b$.

A König's lemma argument shows that because (X, σ) subcompact, it is a G_δ-subspace of $(\mathbb{R}, \hat{\sigma})$, and therefore so is $(Y, \sigma|_Y)$. But then $(Y, \sigma|_Y)$ is subcompact in the light of Proposition 7.3, as required for (a).

The proof of assertion (b) is similar. We use the same notation as in the proof of part (a), and we note that (X, σ) is a G_δ-subspace of $(\mathbb{R}, \hat{\sigma})$. Construct $\hat{\sigma}^S$ by isolating all points of the given set S. Then $\hat{\sigma}^S|_X = (\sigma|_X)^S$ and because $\hat{\sigma} \subseteq \hat{\sigma}^S$ we see that (X, σ^S) is a G_δ-subspace of $(\mathbb{R}, \hat{\sigma}^S)$. Now apply Proposition 7.3 to complete the proof of (b). □

Using the same general kind of techniques, in [9] we proved:

Corollary 7.5 *Suppose $X \subseteq \mathbb{R}$ and τ is a topology on X such that (X, τ) is co-compact. Then (X, τ) is subcompact.*

Outline of Proof: As in the previous corollary, find a GO-topology $\hat{\sigma}$ on $\mathbb{R}$ with $\sigma = \hat{\sigma}|_X$. Let $Z = \mathrm{cl}_{\hat{\sigma}}(X)$ and show that $(Z, \sigma|_Z)$ is a G_δ-subset of $(\mathbb{R}, \hat{\sigma})$. Using co-compactness, show that X is a G_δ-subspace of $(Z, \hat{\sigma}|_Z)$. Then (X, σ) is a G_δ-subspace of $(\mathbb{R}, \hat{\sigma})$ so that Proposition 7.3 completes the proof. □

For GO-spaces (X, σ) on sets $X \subseteq \mathbb{R}$ that are *dense-in-themselves* (i.e., (X, σ) has no isolated points), we have preliminary results relating subcompactness, pseudo-completeness, and domain representability.

Proposition 7.6 *Suppose that σ is a GO-topology on some subset $X \subseteq \mathbb{R}$ and that (X, σ) is dense-in-itself. Then the following are equivalent:*

a) there is a G_δ-subspace S of the usual real line topology such that S is dense in (X, σ) and $(S, \sigma|_S)$ is a subcompact space;

b) (X, σ) is pseudo-complete in the sense of Oxtoby [27];

c) there is a G_δ-subset S of the usual real line such that S is a dense subset of (X, σ).

Many questions remain open concerning GO-spaces constructed on $\mathbb{R}$ and its subsets. For example:

Question 7.7 *Let $X \subseteq \mathbb{R}$.*

a) Suppose τ is a GO-topology constructed on $X \subseteq \mathbb{R}$, and suppose (X, τ) is domain representable. Is (X, τ) subcompact?

b) Characterize those subsets $X \subseteq \mathbb{R}$ that admit some dense-in-itself GO-topology σ such that (X, σ) has one of the Amsterdam properties or is Choquet complete, or is Choquet complete where the non-empty player has a stationary winning strategy.

c) Is it true that if $X \subseteq \mathbb{R}$ and μ is a GO-topology on X such that (X, μ) is Choquet complete, then (X, μ) is domain representable? (Note that the answer is "Yes" if we assume that the non-empty player has a stationary winning strategy [3].)

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