Domain Representability and the Choquet Game in Moore and BCO-spaces

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Abstract: In this paper we investigate the role of domain representability and Scott-domain representability in the class of Moore spaces and the larger class of spaces with a base of countable order. We show, for example, that in a Moore space, the following are equivalent: domain representability; subcompactness; the existence of a winning strategy for player α (= the non-empty player) in the strong Choquet game $Ch(X)$; the existence of a stationary winning strategy for player α in $Ch(X)$; and Rudin completeness. We note that a metacompact Čech-complete Moore space described by Tall is not Scott-domain representable and also give an example of Čech-complete separable Moore space that is not co-compact and hence not Scott-domain representable. We conclude with a list of open questions.

Key Words and Phrases: domain representable, Scott-domain representable, Moore space, base of countable order, strong Choquet game, Choquet completeness, stationary strategy, Čech-complete, Rudin complete, motonically complete base of countable order, co-compact space, normality in Moore spaces.

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1 Introduction

To say that a topological space X is *domain-representable* means that X is homeomorphic to the set of maximal elements of some continuous dcpo endowed with the relative Scott topology. (Definitions appear in Section 2.) We now know that domain representability is stronger than the Baire space property (= each countable intersection of dense open sets is dense) and has close ties with other completeness properties. For example, complete metric spaces and locally compact Hausdorff spaces are domain-representable and, more generally, so are Čech-complete spaces [3] and spaces with one of the "Amsterdam properties," (regular co-compactness, base-compactness, or subcompactness) [4], [1]. In another direction, Martin [17] linked domain representability to Choquet's completeness properties by proving that in any domain representable space X, Player α has a winning strategy in the strong Choquet game $Ch(X)$, and in [4] we provided a partial converse by showing that if a space X has a G_δ-diagonal and if player α has a *stationary* winning strategy in the strong Choquet game $Ch(X)$, then X must be domain-representable.

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Martin's result combines with work of Choquet to show that among metrizable spaces, domain representability is equivalent to Čech completeness and to complete metrizability. In this paper we study the role of domain representability in certain generalized metric spaces, namely Moore spaces and the class of spaces having a base of countable order (BCO) in the sense of Wicke and Worrell [29]. In Section 3, we show that for BCO spaces, domain representability is equivalent to subcompactness, to the existence of a monotonically complete BCO, to the existence of a winning strategy for Player α in Choquet's game $Ch(X)$, and to the existence of a stationary winning strategy for Player α in $Ch(X)$. Further, we show that any domain-representable BCO space can be represented as a G_δ-subset of an ideal domain in the sense of [18].

Because every Moore space has a BCO, the above equivalences also hold for Moore spaces. Furthermore, in the light of Section 3.2.2 in [1], domain representability in a Moore space is equivalent to a Moore space property known as *weak completeness* or as *Rudin completeness*. An example due to Mary Ellen Rudin [12] shows that weak completeness is, as its name suggests, strictly weaker than the classical completeness property in Moore spaces, called *Moore completeness* or simply *completeness*, and it is known that among completely regular Moore spaces, Moore completeness is equivalent to Čech-completeness (see Section 3.2.2 of [1]). Therefore, our results in Section 3 also show that the equivalence of Čech-completeness and domain representability that holds in metrizable spaces will break down in the wider class of Moore spaces, with Čech-completeness being strictly stronger than domain-representability.

As it happens, there is a domain-theoretic property that is stronger than domain-representability, namely Scott-domain-representability. Martin's results in [17] show that any Scott-domain-representable Moore space is completely regular and Moore-complete, and hence Čech-complete [1]. A question in [19] asks whether Scott-domain-representability and Moore completeness are equivalent in the class of completely regular Moore spaces. In Section 4 of this paper we present details of a negative answer to that question. The results in Section 4 were previously announced by the third author in several domain-theory conferences, but the details have not been published.

Throughout this paper, all spaces are assumed to be at least T_3. Relevant definitions, of which there are many, appear in Section 2, below. We want to thank Keye Martin and the referee for suggesting numerous improvements of an earlier draft of our paper.

2 Background

Domain theory: Let $(Q,\sqsubseteq)$ be a partially ordered set. A subset $E \subseteq Q$ is *directed* if for each $e_1,e_2 \in E$, some $e_3 \in E$ has $e_1,e_2 \sqsubseteq e_3$.To say that $(Q,\sqsubseteq)$ is a *dcpo* (= directed-complete-partial-order) means that every non-empty directed subset $E \subseteq Q$ has a supremum in Q, i.e., that there is an upper bound $q \in Q$ for the set E such that $q \sqsubseteq q'$ for every upper bound $q' \in Q$ of the set E. For $a,b \in Q$ we say that $a \ll b$ provided that for any directed set E with $b \sqsubseteq \sup(E)$, some $e \in E$ has $a \sqsubseteq e$. The set of all maximal elements of $(Q,\sqsubseteq)$ will be denoted by $\max(Q)$. Zorn's lemma shows that for each a in a dcpo $(Q,\sqsubseteq)$, some $b \in \max(Q)$ has $a \sqsubseteq b$. To say that a dcpo $(Q,\sqsubseteq)$ is *continuous* means that for each $b \in Q$, the set $\Downarrow(b) = \{a \in Q : a \ll b\}$ is directed and has $b = \sup(\Downarrow(b))$. A *domain* is a continuous dcpo. A domain Q

[1]Consequently, any Scott-domain-representable metrizable space is completely metrizable, and the authors of [20] conjectured that for metrizable spaces, Scott-domain-representability and Čech completeness are equivalent. That conjecture has recently been proved in [16].

is said to be an *ideal domain* if each non-maximal $q \in Q$ has $q \ll q$ [18]. Two elements q_1, q_2 of a domain Q are said to have a common extension in Q if there is some $q \in Q$ with $q_i \sqsubseteq q$ for $i = 1,2$. A domain $(Q, \sqsubseteq)$ is a *Scott domain* if $\sup(\{q_1, q_2\}) \in Q$ whenever q_1, q_2 have a common extension in Q and Q has a minimal element [2]. We say that a set $S \subseteq Q$ is bounded in Q if for some $q \in Q, s \sqsubseteq q$ for each $s \in S$. In a Scott domain Q, every nonempty bounded set has a supremum in Q.

For a domain $(Q, \sqsubseteq)$, the collection $\{\Uparrow(q) : q \in Q\}$ (where $\Uparrow(q) = \{b \in Q : q \ll b\}$) is a basis for a topology on Q known as the *Scott topology*. To say that a space X is *domain-representable* means that for some domain $(Q, \sqsubseteq)$, X is homeomorphic to the space $\max(Q)$ endowed with the relative Scott topology. In case the domain Q is a Scott-domain, we say that X is *Scott-domain representable*.

Topological game theory: The *Choquet game* $Ch(X)$ (sometimes called the "strong Choquet game") in a topological space X is an infinite two-person game that begins when Player α specifies the space X in which the game is to be played. Then Player β responds by choosing a pair (U_1, x_1) where U_1 is open in X and $x_1 \in U_1$. Player α responds by specifying an open set V_1 with $x_1 \in V_1 \subseteq U_1$. Player β then specifies a pair (U_2, x_2) with U_2 open and $x_2 \in U_2 \subseteq V_1$, and Player α responds by choosing an open set V_2 with $x_2 \in V_2 \subseteq U_2$. A *partial play of* $Ch(X)$ is any finite sequence $X, U_1, x_1, V_1, U_2, x_2, V_2, \cdots, U_n, x_n, V_n$ in which $x_k \in V_k \subseteq U_k$ for each $k \geq 1$, and a *play of the game* is an infinite sequence $X, U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ with $x_n \in V_n \subseteq U_n$ for all $n \geq 1$. (Because it will be clear that the game is being played in a given space X, we will henceforth not mention Player α's opening move.)

Player α wins the play $U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ provided $\bigcap\{V_n : n \geq 1\} \neq \emptyset$ (equivalently, $\bigcap\{U_n : n \geq 1\} \neq \emptyset$). A *strategy for Player* α is a sequence $\langle\sigma_n\rangle$ of functions that tell Player α how to specify the set $V_n = \sigma_n(U_1, x_1, \cdots, U_n, x_n)$ with $x_n \in V_n \subseteq U_n$, and $\langle\sigma_n\rangle$ is a *winning strategy for Player* α provided Player α wins any play of $Ch(X)$ in which the function σ_n is used by Player α to specify the sets V_n.

Any locally compact Hausdorff space is an example of a space in which Player α has a winning strategy in the Choquet game. In response to any partial play $(U_1, x_1, \cdots, U_n, x_n)$, Player α could use the rule "Let $\sigma_n(U_1, x_1, \cdots, U_n, x_n)$ be any open set V with $x_n \in V \subseteq \mathrm{cl}(V) \subseteq U_n$ where $\mathrm{cl}(V)$ is compact." This certainly gives a winning strategy for Player α in $Ch(X)$, but more is true. Note that the strategy $\langle\sigma_n\rangle$ in the locally compact example has the special property that Player α's response $\sigma_n(U_1, x_1, \cdots, U_n, x_n)$ really depends only on Player β's most recent move (U_n, x_n) and does not consider the number n or any of the earlier-chosen sets. Any strategy that depends only on the previous move, and not on any part of the earlier history of a partial play, is called a *stationary strategy*. It is easy to see that Player α also has a stationary winning strategy in the Choquet game $Ch(X)$ provided X is a complete metric space. More generally, Porada [22] has shown that Player α has a stationary winning strategy in the Choquet game played on any Čech-complete space, and K. Martin showed in [17] that Player α has a stationary winning strategy in $Ch(X)$ whenever X is representable using a special kind of domain called a Scott domain (defined above).

Some completeness properties: It is well-known that there are metric spaces that are Baire spaces [3] and yet their squares are not. On the other hand, it is known that if X is a complete metric space, then the product space X^κ is a Baire space no matter how large the cardinal κ might be. Over the years, many topological properties have been isolated to explain that phenomenon, all generalizing the notion of completeness in a metric space. Prominent among these are the "Amsterdam properties" introduced by deGroot and his students. The weakest of the Amsterdam properties is subcompactness [10], where we say that a

[2]The restriction that Q have a minimal element is not a major one because we will focus on the set of maximal elements of Q. If Q does not already have a minimal elements, we can simply add one.

[3]i.e., the intersection of countably many dense open subsets of the space is dense in the space

space X is *subcompact* if it has a base $\mathcal{B}$ of open sets such that $\bigcap\mathcal{F} \neq \emptyset$ whenever $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base. (Recall that a collection $\mathcal{F}$ is a *regular filter base* if, given $F_1, F_2 \in \mathcal{F}$, some $F_3 \in \mathcal{F}$ has $\mathrm{cl}(F_3) \subseteq F_1 \cap F_2$.) Such a base is called a *subcompact base* for X. Another Amsterdam property called regular co-compactness was introduced in [11]: a space is *regularly co-compact* if it has a base $\mathcal{B}$ of open sets such that $\bigcap\{\mathrm{cl}(C) : C \in \mathcal{C}\} \neq \emptyset$ whenever $\mathcal{C} \subseteq \mathcal{B}$ and $\{\mathrm{cl}(C) : C \in \mathcal{C}\}$ is centered (= has the finite intersection property). We will use the term *regularly co-compact base* to describe a base with the properties in that definition. Clearly, any regularly co-compact space is subcompact. A notion that is weaker then regular cocompactness is called *cocompactness*. A topological space X is *cocompact* if there is a collection $\mathcal{C}$ of closed subsets of X with the properties that (i) given $x \in U$ where U is open in X, some $C \in \mathcal{C}$ has $x \in \mathrm{Int}(C) \subseteq C \subseteq U$ and (ii) if $\mathcal{D} \subseteq \mathcal{C}$ is centered, then $\bigcap\mathcal{D} \neq \emptyset$.

In the next section we describe various completeness properties in Moore spaces and BCO spaces, and the referee suggested that it would be helpful to readers for us to summarize the relations among the above completeness properties in completely regular spaces. Obviously Scott-domain representable spaces are domain representable. Less obvious is that Scott-domain representable spaces are co-compact ([16] and see Lemma 4.1, below) and that in Scott-domain representable spaces, Player α has a stationary winning strategy in the strong Choquet game [17]. Regularly cocompact spaces are obviously cocompact (but not conversely, as can be seen from the Sorgenfrey line [1]) and subcompact. Subcompact spaces are domain representable, and in a subcompact space Player α has a stationary winning strategy in the strong Choquet game Čech-complete spaces are domain representable, being G_δ-subsets of compact spaces [3], and in any Čech-complete space, Player α has a stationary winning strategy in the strong Choquet game [22].

Moore spaces and their completeness conditions: Moore spaces are a classical generalization of metrizable spaces. A *Moore space* is a T_3-space having a development, i.e., a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of X such that for each point $x \in X$, the collection $\{\mathrm{St}(x, \mathcal{G}(n)) : n \geq 1\}$ is a neighborhood base at x. There are several non-equivalent types of completeness properties in Moore spaces (see Chapter 3 of [1] for details). One is called "weak completeness" or "Rudin completeness" and is defined as follows: a Moore space is Rudin complete if it has a development $\langle \mathcal{G}_n \rangle$ such that

a) $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ for each $n \geq 1$;

b) if $G_n \in \mathcal{G}_n$ and $\mathrm{cl}(G_{n+1}) \subseteq G_n$ for each $n \geq 1$, then $\bigcap\{G_n : n \geq 1\} \neq \emptyset$.

The development with properties (a) and (b) is usually called a *weakly complete development*. A stronger completeness property in Moore spaces is called *Moore completeness* or simply "completeness," and requires that there be a development $\langle \mathcal{G}(n) \rangle$ for X with the property that $\mathcal{G}(n+1)$ refines $\mathcal{G}(n)$ for all n and with the property that $\bigcap\{M_k : k \geq 1\} \neq \emptyset$ whenever $\langle M_k \rangle$ is a nested sequence of nonempty closed sets such that the set M_k is contained in some member of $\mathcal{G}(k)$. It is known (see Section 3.2.2 of [1]) that weak completeness is equivalent to (countable) subcompactness in a Moore space, while Moore completeness is equivalent to (countable) Čech-completeness.

BCO spaces and their completeness conditions: An even broader class than Moore spaces is the class of BCO-spaces [29]. BCO abbreviates "base of countable order," where we say that a base $\mathcal{B}$ for a space is a *base of countable order* if any sequence $\langle B_n \rangle$ of distinct members of $\mathcal{B}$ with $B_{n+1} \subseteq B_n$ is a local base at any point of the set $S = \bigcap\{B_n : n \geq 1\}$. (Note that the set S might be empty.) An equivalent description of BCO spaces is given in [29]:

Theorem 2.1 *A T_3-space X has a BCO if and only if there is a sequence $\mathcal{B}_n$ of bases for X such that whenever the sets $B_n \in \mathcal{B}_n$ have $p \in B_{n+1} \subseteq B_n$, then $\{B_n : n \geq 1\}$ is a local base at p.*

The key difference between the characterization given in Theorem 2.1 and in the original definition is that the sets B_n in Theorem 2.1 are not required to be distinct.

There is a completeness condition associated with BCO-theory: a base $\mathcal{B}$ for a T_3-space is a *monotonically complete BCO* if for any decreasing sequence $\langle B_n \rangle$ of distinct members of $\mathcal{B}$ with $\mathrm{cl}(B_{n+1}) \subseteq B_n$, then the set $T = \bigcap\{B_n : n \geq 1\} \neq \emptyset$ and the collection $\{B_n : n \geq 1\}$ is a neighborhood base at every point of T (so that the set T must be a singleton). Another approach to BCO theory (using sieves of open sets) was introduced in [5], and we will use it in Section 3, below. Gruenhage [15] gives a particularly clear proof of the sieve characterization of spaces with BCOs. Minor modifications of that proof give a related characterization of spaces with a monotonically complete BCO that we will need in the next section.

Theorem 2.2 *A T_3 space (X, τ) has a monotonically complete base of countable order if and only if there is a tree $(T, \sqsubseteq)$ with levels $T_1, T_2, \cdots$ and a function $G : T \to \tau - \{\emptyset\}$ such that:*

a) $\{G(t) : t \in T_1\}$ *covers X;*

b) *if $t \in T_n$ then* $G(t) = \bigcup\{G(t') : t' \in T_{n+1} \text{ and } t \sqsubseteq t'\}$;

c) *if $t_1, t_2, \cdots$ is a branch of T, then the set $S = \bigcap\{G(t_n) : n \geq 1\} \neq \emptyset$ and $\{G(t_n) : n \geq 1\}$ is a local base at each point of S.* $\square$

3 Domain-representability in BCO-spaces

As noted in the Introduction, among metrizable spaces, domain representability, the existence of a winning strategy for Player α in $Ch(X)$, and Čech completeness are mutually equivalent, while in Moore spaces, they are not. Our goal is to study the place of domain representability in the hierarchy of completeness properties in the even larger class of BCO spaces. In this section we present a sequence of lemmas that prove the following result.

Theorem 3.1 *Let X be a T_3-space having a base of countable order. Then the following are equivalent:*

a) X has a monotonically complete BCO;

b) X is subcompact;

c) Player α has a stationary winning strategy in the Choquet game $Ch(X)$;

d) Player α has a winning strategy in the Choquet game $Ch(X)$;

e) X is domain-representable.

In addition, for any domain-representable BCO space X, there is an ideal domain Q such that $\max(Q)$ *is homeomorphic to X and is a G_δ-subset of Q with the Scott topology.*

Parts of Theorem 3.1 are already known. The equivalence of a) and b) is due to Wicke and Worrell [28]. The implication $b) \Rightarrow e)$ is a special case of Theorem 2.1 of [4]. Colleagues have told us that the rest of our equivalences can be obtained using an approach similar to Topsoe's characterization of sieve-completeness using a variant of the game $Ch(X)$. Our goal in this section is to give a more self-contained proof of Theorem 3.1.

Outline of Proof of Theorem 3.1: Lemma 3.2 shows that $a) \Rightarrow b)$ in any T_3-space and Lemma 3.3 shows that $b) \Rightarrow c)$ in any T_3-space. Clearly $c) \Rightarrow d)$ in any space. In Lemma 3.4 we use a result of Chaber, Choban, and Nagami [5] to prove that $d) \Rightarrow a)$ in any T_3-space having a BCO. Thus, items a) through d) are equivalent in any T_3-space having a BCO. To complete the proof of Theorem 3.1, we recall Martin's result [17] that $e) \Rightarrow d)$ in any space, and we use Lemma 3.5 to show that $a) \Rightarrow e)$. The Theorem's final assertion about ideal domains and G_δ-subsets is proved as part of Lemma 3.5. □

Lemma 3.2 *In a T_3-space X, any monotonically complete BCO is a subcompact base.*

Proof: Let $\mathcal{B}$ be a monotonically complete BCO for the T_3-space X, and suppose that $\mathcal{F} \subseteq \mathcal{B}$ is a regular filter base. We must show that $\bigcap \mathcal{F} \neq \emptyset$. If $\mathcal{F}$ has a minimal element (with respect to set-inclusion), there is nothing to prove, so suppose we can choose distinct $B_n \in \mathcal{F}$ with $\mathrm{cl}(B_{n+1}) \subseteq B_n$. Because the sequence B_n was chosen from a monotonically complete BCO, we know that the set $S = \bigcap\{B_n : n \geq 1\} \neq \emptyset$ and that $\{B_n : n \geq 1\}$ is a local base at each point of S. Because X is T_1 it follows that S is a singleton, say $S = \{q\}$ for some $q \in X$. If $q \notin \mathrm{cl}(B)$ for some $B \in \mathcal{F}$, then $B_n \cap B = \emptyset$ for some n and that is impossible because $B, B_n \in \mathcal{F}$ guarantees that some $B' \in \mathcal{F}$ has $\emptyset \neq B' \subseteq B_n \cap B$. Hence $q \in \bigcap \mathcal{F}$. □

Lemma 3.3 *If X is a subcompact T_3-space, then Player α has a winning stationary strategy in the Choquet game $Ch(X)$.*

Proof: Let $\mathcal{B}$ be a subcompact base for X. Given any pair (U,x) with U open and $x \in U$, let $\sigma(U,x)$ be any member $B \in \mathcal{B}$ with $x \in B \subseteq \mathrm{cl}(B) \subseteq U$. If $U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ is a play of $Ch(X)$ in which $V_n = \sigma(U_n, x_n)$, then $\{V_n : n \geq 1\}$ is a regular filter base in $\mathcal{B}$ so that $\bigcap\{V_n : n \geq 1\} \neq \emptyset$, guaranteeing a win for Player α. □

Lemma 3.4 *Suppose that X is a T_3-space with a base of countable order. If Player α has a winning strategy in the Choquet game $Ch(X)$, then X has a monotonically complete base of countable order.*

Proof: Let $\langle \mathcal{B}_n \rangle$ be a sequence of bases for X as described in Theorem 2.1 and let $\langle \sigma_n \rangle$ be a winning strategy for Player α in $Ch(X)$. Define $L_1 = \{(U_1, x_1) : x_1 \in U_1 \in \mathcal{B}_1\}$ and for $(U_1, x_1) \in L_1$ let $G(U_1, x_1) = \sigma_1(U_1, x_1)$. Let

$$L_2 = \{(U_1, x_1, U_2, x_2) : (U_1, x_1) \in L_1,\ x_2 \in U_2 \subseteq G(U_1, x_1) \text{ and } U_2 \in \mathcal{B}_2\}$$

and for $(U_1, x_1, U_2, x_2) \in L_2$ let $G(U_1, x_1, U_2, x_2) = \sigma_2(U_1, x_1, U_2, x_2)$. In general, given L_k and G defined on L_k, we let L_{k+1} be the collection of all $(U_1, x_1, \cdots, U_k, x_k, U_{k+1}, x_{k+1})$ with

$$(U_1, x_1, \cdots, U_k, x_k) \in L_k,\ x_{k+1} \in U_{k+1} \subseteq G(U_1, x_1, \cdots, U_k, x_k) \text{ and } U_{k+1} \in \mathcal{B}_{k+1}.$$

For $(U_1, x_1, \cdots, U_k, x_k, U_{k+1}, x_{k+1}) \in L_{k+1}$ we let

$$G(U_1, x_1, \cdots, U_k, x_k, U_{k+1}, x_{k+1}) = \sigma_{k+1}(U_1, x_1, \cdots, U_k, x_k, U_{k+1}, x_{k+1}).$$

Now let $T = \bigcup\{L_k : k \geq 1\}$ and define a relation $\sqsubseteq$ on T by the rule that if $t_j = (U_1, x_1, \cdots, U_j, x_j) \in L_j$ and $t_k = (V_1, y_1, \cdots, V_k, y_k) \in L_k$, then

$$t_j \sqsubseteq t_k \text{ if and only if } j \leq k \text{ and } (U_i, x_i) = (V_i, y_i) \text{ for } 1 \leq i \leq j.$$

(In other words, $\sqsubseteq$ is end-extension.) Then $(T, \sqsubseteq)$ is a tree and if $t \in T_k$, then the function G satisfies $G(t) = \bigcup\{G(t') : t' \in T_{k+1} \text{ and } t \sqsubseteq t'\}$. To complete the proof, suppose that $t_i \in L_i$ with $t_i \sqsubseteq t_{i+1}$ for each $i \geq 1$, that is, suppose $\langle t_i \rangle$ is a branch of T. Then there is a sequence of pairs (U_i, x_i) such that $t_i = (U_1, x_1, \cdots, U_i, x_i)$. Furthermore if $W_i = G(U_1, x_1, \cdots, U_i, x_i)$ then $W_i = \sigma_i(U_1, x_1, \cdots, U_i, x_i) = G(t_i)$ so that $U_1, x_1, W_1, U_2, x_2, W_2, \cdots$ is a play of the game $Ch(X)$ in which Player α has used the winning strategy $\langle \sigma_k \rangle$. Consequently the set $S = \bigcap\{W_i : i \geq 1\} = \bigcap\{G(t_i) : i \geq 1\} \neq \emptyset$. Let $q \in S$. Because $W_{i+1} \subseteq U_i \subseteq W_i$ we know that $q \in U_i$ for each $i \geq 1$ so that because $U_i \in \mathcal{B}_i$, we know that $\{U_i : i \geq 1\}$ must be a local base at q (remember that $\langle \mathcal{B}_i \rangle$ was chosen using Theorem 2.1). Hence $\{W_i = G(t_i) : i \geq 1\}$ is also a local base at q. Now Theorem 2.2 completes the proof. $\square$

Lemma 3.5 *Suppose X is a T_3-space with a monotonically complete BCO. Then there is a domain Q such that* $\max(Q)$ *is a G_δ-subset of Q (in the Scott topology) and X is homeomorphic to* $\max(Q)$. *In addition, in the terminology of [18], Q is an ideal domain.*

Proof: Because X has a monotonically complete BCO (as defined in Section 2), X has a λ-base, i.e., a base $\mathcal{B}$ such that if $\langle B_n \rangle$ is a strictly decreasing sequence of members of $\mathcal{B}$, then the set $T = \bigcap\{\mathrm{cl}(B_n) : n \geq 1\}$ is non-empty, and if $x \in T$ and if $x \in U$ where U is open in X, some B_n has $B_n \subseteq U$. Because X is T_3, it follows that the set T is a singleton.

Let $Q_0 = \{(B, n) : B \in \mathcal{B}, |B| > 1, n \geq 1\}$ and let $Q_\omega = \{\mathcal{B}(x) : x \in X\}$ where $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$. For $q_1, q_2 \in Q = Q_0 \cup Q_\omega$, we define $q_1 \sqsubseteq q_2$ if and only if one of the following holds:

a) $q_1 = q_2$;

b) $q_i = (B_i, n_i) \in Q_0$ with $n_1 < n_2$, $\mathrm{cl}(B_2) \subseteq B_1$ and $B_1 \neq B_2$;

c) $q_1 = (B_1, n_1) \in Q_0$, $q_2 = \mathcal{B}(y) \in Q_\omega$, and $B_1 \in \mathcal{B}(y)$.

Then $\sqsubseteq$ is a partial order on Q, and $\max(Q) = Q_\omega$. Note that the following prohibited relationship never occurs:

(*) $q_1 \sqsubseteq q_2$ where $q_1 \in Q_\omega$ and $q_2 \in Q_0$.

The rest of the proof involves verifying a sequence of claims. If q is any ordered pair, then for $i = 1, 2$, $\pi_i(q)$ denotes the i^{th} coordinate of q.

<u>Claim 1</u>: $(Q, \sqsubseteq)$ is a dcpo. It will be enough to show that if E is a directed subset of Q that contains no maximal element of itself, and if $\mathcal{F}(E) = \{B \in \mathcal{B} : \exists e \in E \text{ with } \pi_1(e) \subseteq B\}$, then $\mathcal{F}(E) \in Q_\omega$ and $\sup(E) = \mathcal{F}(E)$. Because E contains no maximal element of itself, $E \cap Q_\omega = \emptyset$ so that $E \subseteq Q_0$. Choose distinct $e_i \in E$ with $e_i \sqsubseteq e_{i+1}$ for each i and write $e_i = (B_i, n_i)$. Then the sets B_i are distinct, $\mathrm{cl}(B_{i+1}) \subseteq B_i$ and $n_i < n_{i+1}$. Because $\mathcal{B}$ is a λ-base for X, we know that for some point $x \in X$, $x \in \bigcap\{\mathrm{cl}(B_i) : i \geq 1\} = \bigcap\{B_i : i \geq 1\}$ and that $\{B_i : i \geq 1\}$ is a local base at x.

We claim that $x \in \bigcap\{\pi_1(e) : e \in E\}$. If not, choose $\hat{e}_1 \in E$ with $x \notin \pi_1(\hat{e}_1)$, and choose $\hat{e}_2 \in E - \{\hat{e}_1\}$ with $\hat{e}_1 \sqsubseteq \hat{e}_2$. Then $x \notin \mathrm{cl}(\pi_1(\hat{e}_2))$ so that for some e_i, $\pi_1(e_i) \cap \pi_1(\hat{e}_1) = \emptyset$. That is impossible because

e_i and $\hat{e}_2$ both belong to the directed set E. Therefore, $\bigcap\{\pi_1(e) : e \in E\} = \{x\}$. Consequently, $\mathcal{F}(E) = \mathcal{B}(x) \in Q_\omega$. Once we know that $\mathcal{F}(E) \in Q$, it is clear that $\mathcal{F}(E) = \sup(E)$. Thus, Claim 1 holds.

Claim 2: If $q \in Q_0$ then $q \ll q$. Write $q = (B_0, n_0)$ and suppose that E is a directed set with $(B_0, n_0) \sqsubseteq \sup(E)$. If E contains a maximal element of itself, there is nothing to prove, so assume that no member of E is maximal in E. Then (as in the proof of Claim 1) we choose distinct $e_i = (B_i, n_i) \in E$ with $e_i \sqsubseteq e_{i+1}$, find a point $x \in \bigcap\{\pi_1(e) : e \in E\}$ such that $\{\pi_1(e_i) : i \geq 1\}$ is a local base at x and $\sup(E) = \{B \in \mathcal{B} : \exists e \in E \text{ with } \pi_1(e) \subseteq B\} = \mathcal{B}(x)$. From $(B_0, n_0) \sqsubseteq \mathcal{B}(x)$ we know that $x \in B_0$ so that for some $e_i \in E$, $\pi_1(e_i) \subseteq B_0$. Because $\langle \pi_2(e_j) \rangle$ is a strictly increasing sequence of natural numbers, we may choose $j > i$ with $n_j > n_0$. Then $(B_0, n_0) \sqsubseteq e_j \in E$ as required to prove Claim 2.

Claim 3: If x is a non-isolated point of X, then $\mathcal{B}(x) \ll \mathcal{B}(x)$ is false. Let $E = \{(B, n) : x \in B \in \mathcal{B}\}$. Because x is not isolated and X is T_3, the set E is directed by $\sqsubseteq$ and no member of E is maximal in E. Consequently $\sup(E) = \mathcal{B}(x)$. However, no member $(B, n) \in E$ has $\mathcal{B}(x) \sqsubseteq (B, n)$ because that would involve the relationship prohibited in (*). Consequently, $\mathcal{B}(x) \ll \mathcal{B}(x)$ fails.

Claim 4: If x is an isolated point of X, then $\mathcal{B}(x) \ll \mathcal{B}(x)$. Suppose E is a nonempty directed subset of Q with $\mathcal{B}(x) \sqsubseteq \sup(E)$. Then $\sup(E) = \mathcal{B}(x)$ because $\mathcal{B}(x)$ is a maximal element of Q. We will show that some $e_0 \in E$ has $\mathcal{B}(x) \sqsubseteq e_0$. If E contains a maximal element of itself, we may let e_0 be that maximal element and we have $\mathcal{B}(x) \sqsubseteq \sup(E) = e_0 \in E$, as required. Consider the case where E does not contain a maximal element of itself. Then $E \subseteq Q_0$ and, from Claim 1, we know that $\sup(E) = \{B \in \mathcal{B} : \exists e \in E \text{with } \pi_1(e) \subseteq B\}$. Because x is isolated, we have $\{x\} \in \mathcal{B}(x) = \sup(E)$ so that some $e = (B, n) \in E \subseteq Q_0$ has $B \subseteq \{x\}$. But then $|B| = 1$ and that is prohibited by our definition of Q_0. Consequently, the second case cannot occur and the proof of Claim 4 is complete.

Claim 5: $(Q, \sqsubseteq)$ is continuous. Let $q \in Q_0$. Then Claim 2 shows that $q \in \Downarrow(q)$ so that q is a common extension of any two members of $\Downarrow(q)$ and $q = \sup(\Downarrow(q))$. The same argument applies if $q = \mathcal{B}(x)$ where x is isolated. Finally suppose $q = \mathcal{B}(x)$ where x is not isolated. Then Claim 3 shows that $\Downarrow(q) \subseteq \{(B, n) \in Q_0 : (B, n) \sqsubseteq q\}$ and if $(B, n) \in Q_0$ has $(B, n) \sqsubseteq q$ then Claim 2 gives $(B, n) \ll (B, n) \sqsubseteq q$ so that $(B, n) \in \Downarrow(q)$. Therefore $\Downarrow(q) = \{(B, n) \in Q_0 : x \in B\}$. To see that $\Downarrow(q)$ is directed, suppose $(B_i, n_i) \in \Downarrow(q)$ for $i = 1, 2$. Then $x \in B_1 \cap B_2$ so that because x is not isolated, there is some $y \in (B_1 \cap B_2) - \{x\}$ and then some $B_3 \in \mathcal{B}$ with $x \in B)_3 \subseteq \mathrm{cl}(B_3) \subseteq (B_1 \cap B_2) - \{y\}$. Let $n_3 = n_1 + n_2$. Then $(B_3, n_3) \in \Downarrow(q)$ and $(B_1, n_i) \sqsubseteq (B_3, n_3)$ for $i = 1, 2$. Hence $\Downarrow(q)$ is directed. Then, because $\Downarrow(q)$ is a directed subset of Q_0, Claim 1 shows that

$$\sup(\Downarrow(q)) = \{C \in \mathcal{B} : \exists (B, n) \in \Downarrow(q) \text{ with } B \subseteq C\} = \{C \in \mathcal{B} : \exists B \in \mathcal{B} \text{ with } B \subseteq C\} = \mathcal{B}(x) = q,$$

as required to complete the proof of Claim 5.

Claim 6: The function $h : X \to Q$ given by $h(x) = \mathcal{B}(x)$ is a homeomorphism from X onto $\max(Q)$ where the latter space carries the relative Scott topology. This verification is routine.

Claim 7: The set $\max(Q)$ is a G_δ subset of Q. For each $n \geq 1$, the set $U_n = \bigcup\{\Uparrow((B, n)) : (B, n) \in Q_0\}$ is an open subset of Q, and $\max(X) = \bigcap\{U_n : n \geq 1\}$.

Claim 8: $(Q, \sqsubseteq)$ is an ideal domain in the sense of [18] because Claim 2 shows that every element $q \in Q - \max(Q)$ has $q \ll q$, so that each member of Q is either maximal or compact (in the sense of domain theory). $\square$

Because every Moore space is T_3 and has a BCO, Theorem 3.1 has the following consequence for Moore spaces.

Corollary 3.6 *The following properties of a Moore space X are equivalent:*

a) X has a monotonically complete BCO;

b) X is subcompact;

c) Player α *has a stationary winning strategy in the Choquet game* $Ch(X)$*;*

d) Player α *has a winning strategy in the game* $Ch(X)$*;*

e) X is domain-representable;

f) X is weakly complete.

In addition, if X is a domain-representable Moore space, then there is an ideal domain Q with X homeomorphic to $\max(Q)$ *and* $\max(Q)$ *a* G_δ*-subset of Q.*

Proof: Because a Moore space is T_3 and has a BCO, Theorem 3.1 shows that properties (a) through (e) are equivalent, and the domain Q constructed in 3.1 is an ideal domain with $\max(Q)$ a G_δ-subset of Q. In [1], weak completeness is called "Rudin completeness" and Section 3.2.2 of that paper shows that weak completeness of a Moore space is equivalent to subcompactness of a Moore space. □

Choquet [6] proved that, in a metric space, Player α has a winning strategy if and only if Player α has a stationary winning strategy, and Theorem 3.1 extends that result to a much larger class of spaces. It would be interesting to know the extent to which Theorem 3.1 could be generalized, by finding examples of spaces in which Player α has a winning strategy in $Ch(X)$, but not a stationary winning strategy. The literature is somewhat confusing at this point. A theorem of Galvin and Telgarsky [14] shows that in the game $Ch(X)$, Player I has a winning strategy if and only if Player I has a stationary winning strategy. To understand how that theorem fits with our question, recall that Player I in [14] is the player aiming for an empty intersection, so their Player I is our Player β.

As noted in Section 2, Martin [17] proved that if X is representable by a Scott domain, then Player α has a *stationary* winning strategy in the strong Choquet game on X. Our Corollary 3.6, combined with examples in the next section (of complete Moore spaces that are not Scott-domain representable) provides examples showing the converse of Martin's theorem is false.

4 Scott-domain representable Moore spaces

The previous section ended by showing that a Moore space X is homeomorphic to $\max(Q)$ for some continuous dcpo $(Q, \sqsubseteq)$ if and only if X is weakly complete (also called "Rudin complete"). As noted in Section 2, there is a more restrictive type of domain, called a *Scott domain*, and a more restrictive type of completeness in Moore spaces, called *Moore completeness*. Martin [17] proved that any Scott-domain-representable Moore space must be Moore complete and asked whether the converse is true. The goal of this section is to answer that question in the negative and to present examples of Scott-domain-representable Moore spaces of various types. The results of this section were discussed by the third author in a series of conference talks over several years, but details have not been published.

To see that Scott-domain-representability is more restrictive than domain-representability, recall that Scott-domain representable spaces are all completely regular, while there are many domain-representable spaces that are not even regular. (Consequently, Martin's question, above, about Moore-complete Moore spaces must be restricted to completely regular spaces, because there are well-known examples of regular, Moore-complete Moore spaces that are not completely regular [30].) The following lemma, probably due to Kopperman, Kunzi, and Waszkiewicz [16], states another important property of Scott-domain representable spaces.

Lemma 4.1 *Suppose S is a Scott domain. Let $X = \max(S)$. Then X is cocompact and the collection $\mathcal{C} := \{\uparrow(s) \cap X : s \in S\}$ has the following properties:*

a) members of $\mathcal{C}$ are relatively closed subsets of X

b) if U is an open subset of X and $z \in U$ then for some $C \in \mathcal{C}$ we have $z \in \mathrm{Int}_X(C) \subseteq C \subseteq U$

c) if $\mathcal{D} \subseteq \mathcal{C}$ is a centered collection (i.e., has the finite intersection property), then $\bigcap \mathcal{D} \neq \emptyset$.

Finding a completely regular, Moore-complete (equivalently, Čech-complete) Moore space that cannot be represented as $\max(D)$ for any Scott domain D is our next goal.

Example 4.2 *There is a completely regular, metacompact, non-metrizable, Moore-complete Moore space T that is not Scott-domain representable.*

Proof: Tall introduced a machine for constructing non-metrizable metacompact Moore spaces and used it, in [24], to construct metacompact Moore spaces that are Čech-complete and yet not co-compact (which was defined in Section 2).

To describe Tall's spaces, let Y be any uncountable subset of $\mathbb{R}$ and let $D = \{(p,q) \in \mathbb{Q} \times \mathbb{Q} : q > 0\}$. For each $y \in Y$ choose a sequence $\langle d(y,k) \rangle$ of points of D whose Euclidean distance from $(y,0)$ satisfies $||(y,0) - d(y,k)|| < \frac{1}{k}$. Let $\mathcal{F}$ be the collection of all nonempty finite subsets of Y, and let $D^* = D \times \mathcal{F}$. Let $T = (Y \times \{0\}) \cup D^*$. Topologize T in such a way that each point $(d,F) \in D^*$ is isolated, and so that basic neighborhoods of the point $(y,0) \in T$ have the form $N(y,k) = \{(y,0)\} \cup \{(d(y,j),F) : j \geq k \text{ and } y \in F\}$. Let $\mathcal{N}(k) = \{\{(d,F)\} : (d,F) \in D^*\} \cup \{N(y,k) : y \in Y\}$. Then $\langle \mathcal{N}(k) \rangle$ is a Moore-complete development for T and $\mathcal{N}(k)$ is point-finite. In fact, the only member of $\mathcal{N}(k)$ that contains $(y,0) \in T$ is the set $N(y,k)$. The space T is metacompact [24].

The rest of this proof is devoted to showing that no Scott-domain Q has $\max(Q)$ homeomorphic to T. We argue by contradiction: we will suppose that there is a Scott domain $(Q, \sqsubseteq)$ with T homeomorphic to $\max(Q)$ (we will abuse notation and write $T = \max(Q)$) and we will show that there is a base $\mathcal{B}$ for T with respect to which T is regularly co-compact, contrary to the main result in [24].

Suppose $(Q, \sqsubseteq)$ is a Scott domain that represents the space T. Apply Lemma 4.1 to obtain the collection $\mathcal{C} = \{T \cap \uparrow(q) : q \in Q\}$. Let

$$\mathcal{B} := \{C \in \mathcal{C} : \text{ for some } y \in Y, y \in \mathrm{Int}_T(C) \subseteq C \subseteq N(y,1)\} \cup \{\{x\} : x \text{ is isolated in } T\}.$$

Observe that every point of $N(y,1) - \{y\}$ is isolated in T. Consequently, each member of $\mathcal{B}$ is open in T (as well as closed in T). But then Lemma 4.1 shows that $\mathcal{B}$ is a regularly cocompact base for T, and that is impossible. $\square$

The space described in Example 4.2 is not normal. Under *MA* plus the negation of the Continuum Hypothesis, if one begins with a Q-set Y, then the resulting space is a normal, Čech-complete Moore space that is not Scott-domain representable.

The next result in this section provides another example of a Čech-complete Moore space that is not Scott-domain representable. Unlike the previous example (which was metacompact but not separable), this space is separable but not metacompact.

Example 4.3 *There is a completely regular, Čech-complete, separable Moore space that is not cocompact and hence not Scott-domain representable.*

Proof: Consider ${}^{\omega}\omega$, the set of all functions from ω to itself. For $f, g \in {}^{\omega}\omega$, define $f \leq^* g$ to mean that for some $n \in \omega, f(m) \leq g(m)$ for all $m \geq n$. A set $F \subseteq {}^{\omega}\omega$ is *bounded in* ${}^{\omega}\omega$ if there is some g with $f \leq^* g$ for all $f \in F$. There exist unbounded subsets of ${}^{\omega}\omega$, and any unbounded set F has $\omega_1 \leq |F| \leq 2^{\omega}$. If F is unbounded and $F = \bigcup\{F_n : n \geq 1\}$ then one of the sets F_n is also unbounded in ${}^{\omega}\omega$. For a discussion of unbounded sets, see [7]. For any function $f \in {}^{\omega}\omega$, define $\hat{f}(0) = f(0)$ and for $n \geq 1$ define $\hat{f}(n) = 1 + \max\{f(n), \hat{f}(n-1)\}$. Then $\hat{f}$ is (strictly) increasing and if F is unbounded, then so is $\{\hat{f} : f \in F\}$.

Fix any unbounded subset $F \subseteq {}^{\omega}\omega$ of increasing functions and choose any set X of irrational numbers in $[0,1]$ with $|X| \geq |F|$. Then we may index F using X as $F = \{f_x : x \in X\}$ with repetitions allowed, if necessary.

Let $D_0 = \{0,1\}$ and in general $D_n = \{\frac{j}{2^n} : 0 \leq j \leq 2^n\}$. For each $x \in X$ and each $n < \omega$, there are consecutive points $L(x,n), R(x,n) \in D_n$ with $L(x,n) < x < R(x,n)$.

We split each $x \in X$ into two points x_L and x_R which we think of as being located on the real line exactly where x was located, with $x_L < x_R$. (Technically, we could use the lexicographic product $X \times \{-1,1\}$ and let $x_L = (x,-1), x_R = (x,1)$.)

Let $Z = \{x_s : s \in \{L,R\}, x \in X\} \cup \{(d, 2^{-k}, j) : k < \omega, d \in D_k, j < \omega\}$. We topologize Z by isolating every point $(d, 2^{-k}, j)$ and by using the sets

$$B_n(x_L) = \{x_l\} \cup \{(L(x,k), 2^{-k}, j) : k \geq n, j \geq f_x(k)\}$$

and

$$B_n(x_R) = \{x_R\} \cup \{(R(x,k), 2^{-k}, j) : k \geq n, j \geq f_x(k))\}$$

as basic neighborhoods of the points x_L and x_R respectively. It is easy to see that this topology makes Z into a separable Moore space that is Moore-complete and hence Čech complete.

For contradiction, suppose that Z is co-compact with respect to some collection $\mathcal{C}$ of closed sets (i.e., $\mathcal{C}$ has properties a), b), and c) in Lemma 4.1). For each $x \in X$ choose sets $G(x,L), G(x,R) \in \mathcal{C}$ with

$$x_L \in \mathrm{Int}_Z(G(x,L)) \subseteq G(x,L) \subseteq B_1(x_L)$$

and

$$x_R \in \mathrm{Int}_Z(G(x,R)) \subseteq G(x,R) \subseteq B_1(x_R).$$

There are integers j,k with $B_j(x_L) \subseteq G(x,L) \subseteq B_1(x_L)$ and $B_k(x_R) \subseteq G(x,R) \subseteq B_1(x_R)$. Hence there is an integer $n(x) = j + k$ with

$$B_{n(x)}(x_L) \subseteq G(x,L) \subseteq B_1(x_L)$$

and

$$B_{n(x)}(x_R) \subseteq G(x,R) \subseteq B_1(x_R).$$

Define $X_k = \{x \in X : n(x) = k\}$ and $F_k = \{f_x : x \in X_k\}$. Then $X = \bigcup\{X_k : k < \omega\}$ so that $F = \bigcup\{F_k : k < \omega\}$. Consequently, there is some $n_0 < \omega$ such that the set F_{n_0} is unbounded. Then there is some $m > n_0$ (in fact, infinitely many) with the property that $\{f_x(m) : x \in X_{n_0}\}$ is an unbounded set of positive integers. Choose $x_i \in X_{n_0}$ with the property that $f_{x_i}(m) > i$. Observe that

(*) if S is any infinite subset of $\{x_i : i < \omega\}$, then $\{f_s(m) : s \in S\}$ is an unbounded set of positive integers.

This fact will be needed in the proof of the second claim, below.

Because the set $D_m = \{\frac{j}{2^m} : 0 \leq j \leq 2^m\}$ is finite, there are consecutive members $e < e'$ of D_m with the property that the set $W = [e,e'] \cap \{x_i : i < \omega\}$ is infinite. Consider the sets $D_{m+1}, D_{m+2}, \cdots$. There is a first $M > m$ such that D_M has consecutive points $a < b < c$ such that both $W \cap [a,b]$ and $W \cap [b,c]$ are nonempty and one is infinite. Define

$$\mathcal{G} = \{G(x,R) : x \in W \cap [a,b]\} \cup \{G(y,L) : y \in W \cap [b,c]\}.$$

Then $\mathcal{G} \subseteq \mathcal{C}$.

For the rest of the proof, assume that $W \cap [a,b]$ is infinite and $W \cap [b,c] \neq \emptyset$. The other case is analogous. Because $W \subseteq X_{n_0}$, we have the relations

$$B_{n_0}(x_R) \subseteq G(x,R) \subseteq B_1(x_R) \text{ for each } x \in W \cap [a,b]$$

and

$$B_{n_0}(y_L) \subseteq G(y,L) \subseteq B_1(y_L) \text{ for each } y \in W \cap [b,c].$$

<u>Claim 1</u>: The collection $\mathcal{B} = \{B_{n_0}(x_R) : x \in W \cap [a,b]\} \cup \{B_{n_0}(y_L) : y \in W \cap [b,c]\}$ is centered (= has the finite intersection property) and therefore so is $\mathcal{G}$. Recall that $a < b < c$ are consecutive points of D_M. Therefore if $x \in W \cap [a,b]$ and $y \in W \cap [b,c]$ then $R(x,M) = b = L(y,M)$. Because $M > n_0$ it follows that if $x \in W \cap [a,b]$, the set $B_{n_0}(x_R)$ contains all points $(b, 2^{-M}, j)$ for $j > f_x(M)$. Similarly, if $y \in W \cap [b,c]$, the set $B_{n_0}(y_L)$ contains all points $(b, 2^{-M}, j)$ for $j > f_y(M)$. Provided we consider only a finite number of points $x \in W \cap [a,b]$ and only a finite number of points $y \in W \cap [b,c]$ we obtain points $(b, 2^{-M}, j)$ that belong to each of finitely many sets $B_{n_0}(x_R)$ and $B_{n_0}(y_L)$. Therefore the collection $\mathcal{B}$ is centered, as claimed. Hence so is $\mathcal{G}$.

<u>Claim 2</u>: The collection $\mathcal{B}^* = \{B_1(x_R) : x \in W \cap [a,b]\} \cup \{B_1(y_L) : y \in W \cap [b,c]\}$ has $\bigcap \mathcal{B}^* = \emptyset$ and therefore $\bigcap \mathcal{G} = \emptyset$. For contradiction, suppose that $\bigcap \mathcal{B}^* \neq \emptyset$. Observe that no point x_R or x_L belongs to $\bigcap \mathcal{B}^*$ so that some point of the form $(d, 2^{-k}, j)$ is in $\bigcap \mathcal{B}^*$. Then $d \in D_k$.

Either $k < M$ or $k \geq M$. Suppose $k < M$. The definition of M guarantees that there are consecutive points $u < v$ of D_k with $W \subseteq [u,v]$. Choose $x \in W \cap [a,b]$ and $y \in W \cap [b,c]$. Then $x, y \in [u,v]$ so that $R(x,k) = v$ and $L(y,k) = u$. However, $(d, 2^{-k}, j) \in B_1(x_R)$ so that $d = R(x,k)$ and $(d, 2^{-k}, j) \in B_1(y_R)$ so that $d = L(y,k)$ showing that $u = v$, which is false. Therefore $k < M$ is impossible. Next consider the case where $k \geq M$. For each $x \in W \cap [a,b]$ the fact that $(d, 2^{-k}, j) \in B_1(x_R)$ guarantees that $d = R(x,k)$ and $j \geq f_x(k) \geq f_x(M) \geq f_x(m)$ (because each f_x is increasing). Because the set $W \cap [a,b]$ is an infinite subset of $\{x_i : i < \omega\}$, it follows from assertion (*) above that $\{f_x(m) : x \in W \cap [a,b]\}$ is an unbounded set of positive integers, and hence so is $\{f_x(k) : x \in W \cap [a,b]\}$. But that is false, because $f_x(k) \leq j$.

Claims 1 and 2 show that the subcollection $\mathcal{G}$ of $\mathcal{C}$ is centered and has empty intersection. Now apply Lemma 4.1 to conclude that Z is not Scott-domain representable. □

Clearly one can obtain separable Moore spaces with additional special properties related to normality by carefully choosing the set X in Example 4.3. In [13], it is shown that there is a model of set theory in which there exists a Q-set concentrated on the rationals. Hence, in this model there exists a Q-set Y and a non λ-set of the same cardinality κ. The existence of the non λ-set implies there exists an unbounded set of functions F from ω to ω of cardinality κ. Now, if we use the set Y instead of X in Example 4.3, in a manner similar to that of the examples in [23], the resulting space is a consistent example of a countably paracompact, non-normal, separable, Čech-complete Moore space that is not Scott-domain representable. However, in [26] Fleissner announced that under the continuum hypothesis, each countably paracompact, separable Moore space is metrizable. Hence under CH, each countably paracompact, separable, Čech-complete Moore space is Scott-domain representable. Thus, we have

Corollary 4.4 *The statement that each countably paracompact, separable, Čech-complete Moore space is Scott-domain representable is independent of and consistent with ZFC.*□

We close this section by showing that many of the classical examples in Moore space theory are Scott-domain representable.

Proposition 4.5 *Suppose X is a Moore space that can be written as $X = D \cup K$ where D is a closed, discrete subset of X and each point of K is isolated. Suppose that for each $x \in D$ there is an open set $U(x)$ such that*

a) $U(x) \cap D = \mathrm{cl}(U(x) \cap D = \{x\}$, and

b) if x and y are distinct points of D, then $U(x) \cap U(y)$ is finite.

Then X is representable as $\max(D)$ where D is an algebraic Scott domain.

Proof: Using the sets $U(x)$ we can find a development $\mathcal{G}(n) = \{g(n,x) : x \in X\}$ for X such that

1) $\bigcap\{\bigcup\{g(n,x) : x \in D\} : n \geq 1\} = D$

2) if $x \in K$ then $g(n,x) = \{x\}$ for each $n \geq 1$;

3) for each x and each n, $x \in g(n,x) \subseteq U(x)$;

4) for each $m,n \geq 1$, if $x \neq y$ are points of D, then $g(m,x) \cap g(n,y)$ is finite.

For each $n \geq 1$, let

$$P_n = \{(g_n(x),n) : x \in X\} \cup \{(F,n) : F \subseteq K,\ |F| < \omega \text{ and } F \subseteq g(n,x) \text{ for some } x \in X\}.$$

Let $Q_0 == \bigcup\{P_n : n \geq 1\}$ and $Q_\omega = X \times \{\omega\}$, and $Q = Q_0 \cup Q_\omega$. For $q_1, q_2 \in Q$ define $q_1 \sqsubseteq q_2$ to mean that one of the following holds:

i) $q_1 = q_2$;

ii) $q_1 = (r,m) \in P_m$ and $q_2 = (s,n) \in P_n$ with $s \subseteq r$ and $n \leq m$;

iii) $q_1 = (r,n) \in P_n$ and $q_2 = (x,\omega)$ with $x \in X$, and $x \in r$.

Then $\sqsubseteq$ is a partial order on Q, and $\max(Q) = \{(x,\omega) : x \in X\}$.

Consider any non-empty directed subset $E \subseteq Q$ that contains no maximal element of itself. Then $E \subseteq Q_0$ and either

a) there is some $x \in K$ such that all but finitely many elements of E are of the form $(\{x\},n)$ for $n < \omega$; or

b) there is some $x \in D$ such that $E \subseteq \{g(n,x) : n \geq 1\}$.

In either case, $\sup(E) = (x,\omega) \in Q_\omega$. Consequently, Q is a dcpo.

Given that any directed subset $E \subseteq Q$ either contains a maximal elements of itself, or else satisfies (a) or (b), it is easy to prove that if $q \in Q_0$, then $q \ll q$ and that if $(x,\omega) \in Q_\omega$ then $R(x,\omega) \ll (x,\omega)$ is false. It follows that for any $q \in Q$, $\Downarrow(q) = \{p \in Q_0 : p \sqsubseteq q\}$ so that $\Downarrow(q)$ is directed and has q as its supremum, so that Q is a continuous dcpo. Next, observe that if two non-maximal elements $q_i = (R_i,n_i) \in Q_0$ for $i = 1,2$ have a common extension in Q, then $(R_1 \cap R_2, \max(n_1,n_2)) \in Q$ is the supremum of the set $\{q_1,q_2\}$. It follows that Q is a Scott domain. Finally, note that $\max(Q) = Q_\omega$ and the function $h(x) = (x,\omega)$ is a homeomorphism from X onto $\max(Q)$, so the proof is complete. $\square$

Example 4.6 *Isbell's space Ψ [9] is a Moore space that is non-metrizable, pseudo-compact, Moore-complete, and separable, and is Scott-domain representable.*

Proof: Neighborhoods of non-isolated points in Ψ are constructed using members of a maximal almost-disjoint family of subsets of ω. Hence Proposition 4.5 shows that Ψ is Scott-domain-representable. (Alternatively, note that Ψ is a locally compact Hausdorff space, and that any locally compact Hausdorff space is Scott-domain-representable.) $\square$

Example 4.7 *Heath's V-space is a non-metrizable, metacompact Moore space that is homeomorphic to the space of maximal elements of some Scott domain.*

Proof: Heath's V-space is the set $X = \mathbb{R} \times [0,\infty)$ topologized in such a way that each point (x,t) with $t > 0$ is isolated and such that the collection $\mathcal{V}(x) = \{V(x,k) : k \geq 1\}$ is a neighborhood base at $(x,0)$, where $V(x,k) = \{(y,t) \in X : t = \pm(y-x) \leq \frac{1}{k}\}$. Geometrically, the set $V(x,k)$ is a letter V that touches the x-axis only at $(x,0)$ and whose arms are parts of ± 45-degree lines through $(x,0)$. It is easy to see that X is a non-metrizable, metacompact Moore space and that the space satisfies the hypotheses of Proposition 4.5.

Remark 4.8 *We note that if there is a Q-set $S \subseteq \mathbb{R}$ and if we use only those points $(x,0)$ for $x \in S$, then the resulting Heath V-space is non-metrizable, normal, metacompact, and Scott-domain-representable.*

Remark 4.9 *In studying Moore spaces that are regularly co-compact or Scott-domain-representable, it is important to realize that such a space can have one base $\mathcal{B}$ that is regularly co-compact and also another base $\mathcal{C}$ with the property that no subcollection $\mathcal{C}' \subseteq \mathcal{C}$ can be a regularly co-compact base for the space. An example of this type appears in [2].*

5 Open Questions

Many questions remain open about the relation between domain-representability and previously defined types of Baire-category completeness.

Question 5.1 *Find an example of a T_3-space that is domain-representable but not subcompact. (In the light of our results in Section 3, such a space cannot be a Moore space, and cannot have a BCO.)*

Question 5.2 *Find an example of a T_3, domain-representable space X such that Player α does not have a stationary winning strategy in the Choquet game $Ch(X)$ (even though Player α would have a winning non-stationary strategy in $Ch(X)$ in the light of Martin's results in [17]). (We remark that the paper of Galvin and Telgarsky [14] did not settle this question, because their Player I is our player β.)*

Question 5.3 *Find an example of a T_3-space X in which Player α has a stationary winning strategy in the Choquet game $Ch(X)$ and yet X is not domain representable.*

Question 5.4 *Characterize Scott-domain representability in the class of Moore spaces. (Kopperman, Kunzi, and Waszkiewicz [16] have characterized Scott-domain representability among completely regular spaces using a variant of de Groot's co-compactness, and Miškin [21] has characterized related properties (regular co-compactness and base-compactness) in Moore spaces, but we are not aware of any Moore-space characterization of Scott-domain representability.)*

Question 5.5 *In ZFC, is it true that every normal, separable Čech-complete Moore space must be Scott-domain-representable? (Compare Corollary 4.4.)*

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