Domain-Representability of Certain Complete Spaces

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Abstract

In this paper we show that three major classes of topological spaces are domain-representable, i.e., homeomorphic to the space of maximal elements of some domain (=continuous dcpo) with the relative Scott topology. The three classes are: T_3 subcompact spaces, strongly α -favorable spaces with a G_{δ} -diagonal or with a base of countable order, and complete quasi-developable T_3 -spaces. It follows that any regular space with a monotonically complete base of countable order (in the sense of Wicke and Worrell) is domain-representable, as is any space with exactly one limit point. (In fact, any space with exactly one limit point is domain representable using a Scott domain.) The result on strongly α -favorable spaces with a G_{δ} -diagonal can be used to show that spaces such as the Sorgenfrey line, the Michael line, the Moore plane, the Nagata plane, and Heath's V-space are domain-representable, and to show that a domain-representable space can be Hausdorff but not regular.

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1 Introduction

Spaces that are homeomorphic to the space of maximal elements of some domain (= continuous dcpo) with the relative Scott topology are said to be *domain representable*. The idea of representing a topological space as the set of maximal elements of a special kind of poset is borrowed from theoretical computer science [12]. But whatever its origin, the idea has considerable utility in the study of completeness properties associated with *Baire spaces*, i.e. spaces in which the intersection of countably many dense open sets is dense.

Classical theorems [12] assert that any locally compact Hausdorff space, and any completely metrizable space, must be domain-representable. Martin and Reed [11] generalized that second result, proving that any complete Moore space is domain-representable. In an earlier paper [3] we unified those three results by showing that any Čech-complete space is domain-representable.

In this paper we show that all members of certain other broad classes of spaces are domain-representable. In Section 3 we show that any subcompact space (in the sense of J. de Groot [7]) is domain-representable. One immediate consequence is that any subspace of any T_1 -space with exactly one non-isolated point must be domain-representable. (In fact, any such space is Scott-domain representable.) Another consequence is that any T_3 -space with a monotonically complete base of countable

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order [15] is domain-representable. In Section 4 we show that any strongly α -favorable space with a G_{δ} -diagonal is domain-representable, as is any strongly α -favorable space with a base of countable order. We use that result to show that many familiar examples in topology are domain-representable and to show that a domain-representable space can be Hausdorff but not regular. In Section 5 we show that any complete quasi-developable T_3 -space is domain-representable, thereby generalizing the result of Martin and Reed mentioned above.

Topological terms are defined in subsequent sections, as needed. We review basic domain theory definitions here. A *poset* is a partially ordered set. Recall (see [12]) that a *dcpo* is a poset (Q, \sqsubseteq) in which each non-empty directed set *E* has a supremum (= the least of all upper bounds of *E*) in *Q*. For $a, b \in Q$, to write $a \ll b$ means that whenever *E* is a directed set with $b \sqsubseteq \sup(E)$ then $a \sqsubseteq e$ for some $e \in E$. The set $\{a \in Q : a \ll b\}$ is denoted by $\Downarrow(b)$, and to say that *Q* is *continuous* means that for each $b \in Q$, the set $\Downarrow(b)$ is directed and $b = \sup(\Downarrow(b))$. A *domain* is a continuous dcpo. In case $a \in Q$ has $a \ll a$ we say that *a* is a *compact element* of *Q*.

If Q is a domain, then the collection of all sets of the form $\Uparrow(a) = \{b \in Q : a << b\}$ is a basis for a topology called the *Scott topology* on Q. The set of all maximal elements of Q is denoted by $\max(Q)$, and to say that a space X is *domain-representable* means that there is a continuous dcpo Q with X homeomorphic to the space $\max(Q)$ with the relative Scott topology.

The authors want to thank the referee whose comments substantially improved our proofs, as explained in Section 2.

2 An ideal completion theorem for any poset

In an earlier version of this paper, we began with a space X that had some special structure and used X to define a poset Q which we proved to be a domain, and then we showed that X is homeomorphic to the set of maximal elements of the domain Q. In each case, the longest part of our argument was the proof that Q is a domain, a proof that involved the same set of assertions in each case, with different arguments depending upon the special structure assumed for X. The referee pointed out that we were repeatedly re-proving special cases of a well-known theorem in domain theory, namely:

Theorem 2.1 Let (P, \sqsubseteq) be any nonempty poset. Then there is a domain $\mathbb{I}(P)$ that contains an orderisomorphic copy of (P, \sqsubseteq) as the set of its compact elements. \Box

Our proofs in Sections 3,4, and 5 exploit Theorem 2.1. Given *X*, we define a poset P = P(X) which is not a domain, invoke Theorem 2.1 to create the domain $\mathbb{I}(P)$, and then show that the set of maximal elements of $\mathbb{I}(P)$ (with the relative Scott topology) is homeomorphic to *X*. For that last step, we need a characterization of the maximal elements of $\mathbb{I}(P)$, something that follows from the proof of Theorem 2.1 given, for example, in Proposition I-4.10 of [5]). We outline that proof here.

By an *ideal* of the poset (P, \sqsubseteq) we mean a nonempty directed set J with the property that if $x \sqsubseteq y \in J$, then $x \in J$. For any $y \in P$, the set $J(y) := \{x \in P : x \sqsubseteq y\}$ is an ideal of P, called a *fixed ideal*. There may be ideals of P that are not fixed. Let $\mathbb{I}(P)$ be the set of all ideals of P, and partially order $\mathbb{I}(P)$ by inclusion. For a non-empty directed subset \mathcal{I} of $\mathbb{I}(P)$, it is easy to show that $\bigcup \mathcal{I} \in \mathbb{I}(P)$, that $\sup(\mathcal{I}) = \bigcup \mathcal{I}$, and that $(\mathbb{I}(P), \subseteq)$ is a dcpo. It is easy to see that a fixed ideal (defined above) J(x)satisfies $J(x) \ll J(x)$ and that if J, K are any ideals of P, then $J \ll K$ if and only if there is some $y \in K$ with $J \subseteq J(y) \subseteq K$ because $K = \sup\{J(y) : y \in K\}$. It follows that $\mathbb{I}(P), \subseteq)$ is a domain whose compact elements are the fixed ideals J(y). It is easy to check that the correspondence $y \to J(y)$ is an order-isomorphism from (P, \sqsubseteq) onto the set of all compact elements of the domain $(\mathbb{I}(P), \subseteq)$.

Note that the maximal elements of the domain $(\mathbb{I}(P), \subseteq)$ are exactly the maximal ideals of the poset (P, \sqsubseteq) and that if *M* is any such maximal ideal, then the collection $\{\Uparrow(J(y)) \cap \max(\mathbb{I}(P)) : y \in M\}$ is a neighborhood base for *M* in the subspace $\max(\mathbb{I}(P))$ with the relative Scott topology.

Our proofs in Sections 3,4, and 5 all follow a general pattern. In each section we will use a special structure associated with a topological space X to define a poset (P, \sqsubseteq) . Then we will invoke Theorem 2.1 to obtain the domain $\mathbb{I}(P)$. The heart of each proof involves showing that every maximal ideal of P can be associated with a point of the space X in a natural 1-1 way, and that the resulting 1-1 correspondence is the required homeomorphism. This is not automatic.

The remaining sections of the paper are arranged in a kind of increasing order of difficulty. Both the poset and partial order used in Section 3 (for subcompact T_3 -spaces) are natural and straightforward. The partial order used is Section 4 (for strongly α -favorable spaces with a G_{δ} -diagonal) is one step more complicated, and both the poset and the partial order used in Section 5 (for complete quasi-developable T_3 spaces) are even more complicated.

3 Subcompact spaces

A collection \mathcal{R} of subsets of a space X is said to be *regular* if, given $R_1 \in \mathcal{R}$, some $R_2 \in \mathcal{R}$ has $cl(R_2) \subseteq R_1$. A *filter base* is any collection \mathcal{F} of non-empty subsets of X with the property that, given $F_1, F_2 \in \mathcal{F}$ some $F_3 \in \mathcal{F}$ has $F_3 \subseteq F_1 \cap F_2$. A base \mathcal{B} for a space X is a *(countably) subcompact base* for $X \notin \mathcal{B}$ and if whenever $\mathcal{R} \subseteq \mathcal{B}$ is a (countable) regular filter base, then $\bigcap \mathcal{R} \neq \emptyset$. Subcompact spaces, i.e., spaces having a subcompact base for their topology, were introduced and studied by de Groot [7] as a class of Baire spaces with enough additional structure to be very stable under topological operations such as formation of arbitrary products (see also [1]).

Theorem 3.1 If X is a subcompact T_3 -space, then X is representable as the space of maximal elements of some continuous dcpo with the relative Scott topology.

Proof: Let $P = \mathcal{B}$ be a subcompact base for *X* and for $B, C \in P$ define $B \sqsubseteq C$ to mean either that B = C or that $cl_X(C) \subseteq B$. Then (P, \sqsubseteq) is a poset, so that Theorem 2.1 gives us the domain $(\mathbb{I}(P), \subseteq)$ whose members are the ideals of (P, \sqsubseteq) . We now turn to the study of the maximal ideals of *P*.

Claim 1: For each $x \in X$ let $N(x) := \{C \in \mathcal{B} : x \in C\}$. Then $\max(\mathbb{I}(P)) = \{N(x) : x \in X\}$. First we will show that if M is a maximal ideal of P, then $\bigcap M$ is a singleton subset of X, say $\bigcap M = \{x\}$, and then M = N(x). Consider any maximal ideal M of P. Note that either M contains a maximal element of itself (with respect to the partial order \sqsubseteq of P), in which case $\bigcap M \neq \emptyset$, or else M has no such element, and in that case M is a regular filter in the base \mathcal{B} (see the above definition of a subcompact base) so that $\bigcap M \neq \emptyset$. Hence any maximal ideal M of P has $\bigcap M \neq \emptyset$. Fix any $x \in \bigcap M$ and observe that N(x), defined above, is an ideal of (P, \sqsubseteq) (regularity is used here) with $M \subseteq N(x)$. Maximality of Myields M = N(x) as claimed. Because X is known to be T_1 there can be only one point of $\bigcap M$ so that $\max(\mathbb{I}(P) \subseteq \{N(x) : x \in X\}$. Finally consider any N(x). Zorn's lemma shows that there is a maximal ideal M of P with $N(x) \subseteq M$ and the first part of this argument shows that M = N(y) for some $y \in X$. Then the fact that X is a T_1 -space combines with $N(x) \subseteq M \subseteq N(y)$ to give x = y so that N(x) = M is a maximal ideal of P. This completes the proof of Claim 1. *Claim* 2: The function $h: X \to \max(I(P))$ given by $h(x) = N(x) = \{C \in \mathcal{B} : x \in C\}$ is a homeomorphism from X onto $\max(\mathbb{I}(P))$. The argument in Claim 1 shows that *h* is both 1-1 and onto the set $\max(\mathbb{I}(P))$. To verify continuity, fix $x \in X$. Then $M := h(x) = \{C \in \mathcal{B} : x \in C\}$ is a maximal ideal of *P*, and basic Scott neighborhoods of h(x) = M have the form $\uparrow(J(C_0))$ where C_0 is some fixed member of *M*, i.e., some fixed member of \mathcal{B} with $x \in C_0$. We claim that $h[C_0] \subseteq \uparrow(J(C_0))$. For suppose $y \in C_0$. To show that $h(y) \in \uparrow(J(C_0))$ we must show that $J(C_0) \ll h(y)$. Because $J(C_0) \ll J(C_0)$ it is enough to show that $J(C_0) \subseteq h(y)$, so consider any $D \in J(C_0)$. Then in the poset $P = \mathcal{B}$ we have $D \sqsubseteq C_0$ showing that $y \in C_0 \subseteq \operatorname{cl}_X(C_0) \subseteq D$. Hence $y \in D \in \mathcal{B}$ so that $D \in h(y)$, as required. Therefore, *h* is continuous.

Next, *h* is an open mapping from *X* onto $\max(\mathbb{I}(P))$ with the relative Scott topology. Let *U* be any open subset of *X* and suppose $x \in U$. We must show that h[U] is a relative Scott neighborhood of h(x). To that end, use regularity of *X* to find $C_1 \in \mathcal{B}$ with $x \in C_1 \subseteq \operatorname{cl}_X(C_1) \subseteq U$. It is clear that $h(x) \in$ $\uparrow(J(C_1))$ and we claim that $\uparrow(J(C_1)) \cap \max(\mathbb{I}(P)) \subseteq h[U]$. For suppose $M \in \uparrow(J(C_1)) \cap \max(\mathbb{I}(P))$. From Claim 1, we know that for some $y \in X, M = N(y) = \{D \in \mathcal{B} : y \in D\}$. If $y \in U$ then $M = h(y) \in$ h[U] and we are done. If $y \notin U$, then $y \notin \operatorname{cl}_X(C_1)$ so there is some D_1 in the base \mathcal{B} with $y \in D_1$ and $D_1 \cap C_1 = \emptyset$. Because $M \in \uparrow(J(C_1))$ we know that $J(C_1) \subseteq M$, so that both of C_1 and D_1 belong to M, and that is impossible because $C_1 \cap D_1 = \emptyset$. Therefore Claim 2 is established and the proof of Theorem 3.1 is complete. \Box

Theorem 3.1 can be applied to other classes of spaces. Recall that a base \mathcal{B} for a space X is said to be a *base of countable order* (BCO) [16] if whenever $\langle B_i \rangle$ is a decreasing sequence of distinct members of \mathcal{B} with $\bigcap \{B_n : n \ge 1\} \neq \emptyset$, then $\{B_n : n \ge 1\}$ is a local base at each point of $\bigcap \{B_n : n \ge 1\}$. (It follows that if X is Hausdorff, then the set $\bigcap \{B_n : n \ge 1\}$ can have at most one point for any such sequence $\langle B_n \rangle$.) The completeness property associated with BCO-theory is called *monotonic completeness* [15]: a BCO \mathcal{B} is monotonically complete if $\bigcap \{cl(B_n) : n \ge 1\} \neq \emptyset$ whenever $B_n \in \mathcal{B}$ has $B_{n+1} \subseteq B_n$ for each n. If it happens that $cl(B_{n+1}) \subseteq B_n$ for each n, then $\bigcap \{B_n : n \ge 1\} \neq \emptyset$.

Lemma 3.2 Any T_3 - space with a monotonically complete BCO is subcompact.

Proof: Let \mathcal{B} be a monotonically complete base for X, and consider any regular filter base $\mathcal{F} \subseteq \mathcal{B}$. For contradiction, suppose $\bigcap \mathcal{F} = \emptyset$. Then \mathcal{F} is infinite and has no minimal element (with respect to set inclusion), so that we may choose distinct $B_n \in \mathcal{F}$ with $cl(B_{n+1}) \subseteq B_n$. Then there is a point $p \in X$ such that $\bigcap \{B_n : n \ge 1\} = \{p\}$. However, $p \notin \bigcap \mathcal{F}$ so that some $B_0 \in \mathcal{F}$ has $p \notin B_0$. Then $\bigcap \{B_k : k \ge 0\} = \emptyset$. Now choose any $\hat{B}_1 \in \mathcal{F}$ with $cl(\hat{B}_1) \subseteq B_0 \cap B_1$ and then, given $\hat{B}_k \in \mathcal{F}$ choose $\hat{B}_{k+1} \in \mathcal{F}$ with $cl(\hat{B}_{k+1}) \subseteq B_{k+1} \cap \hat{B}_k$. Because \mathcal{B} is monotonically complete, we know that $\bigcap \{\hat{B}_k : k \ge 1\} \neq \emptyset$. However, $\bigcap \{cl(\hat{B}_k) : k \ge 1\} \subseteq \bigcap \{B_k : k \ge 0\} = \emptyset$, and that is impossible. Therefore $\bigcap \mathcal{F} \neq \emptyset$ as claimed. \Box

Corollary 3.3 Any T_3 space with a monotonically complete BCO is domain-representable. \Box

The following result was originally obtained by G.M. Reed [14] and can also be proved from Theorem 3.1. Recall that a Moore space X is *weakly complete* (also called *Rudin complete* in [1]) if there is a development $\langle \mathcal{G}(n) \rangle$ for X such that:

- a) for each n, $\mathcal{G}(n+1) \subseteq \mathcal{G}(n)$;
- b) if for each $n, G(n) \in \mathcal{G}(n)$ satisfies $cl(G(n+1)) \subseteq G(n)$, then $\bigcap \{G(n) : n \ge 1\} \neq \emptyset$.

For more on different types of completeness in Moore spaces, see Section 4.

Corollary 3.4 Any weakly complete Moore space is representable as the space of maximal elements of some continuous dcpo, with the relative Scott topology.

Proof: According to the Lemma in (3.2.2) of [1], weakly complete Moore spaces are precisely subcompact Moore spaces. Now apply Theorem 3.1.

Alternatively, use Theorem 1 of [15] to show that any weakly complete Moore space has a monotonically complete BCO, and then apply Corollary 3.3. \Box

A continuous dcpo Q is called a *Scott domain* if it has the additional property that whenever two elements $q_1, q_2 \in Q$ have some common extension, then they have a least common extension $\sup(q_1, q_2)$. It is natural to wonder which domain-representable spaces are actually representable as the space of maximal elements of some Scott domain. For example, Theorem 3.1 cannot be strengthened to assert that any subcompact T_3 space is homeomorphic to the space of maximal elements of some Scott domain, then X is Moore space X can be represented as the space of maximal elements of some Scott domain, then X is Moore complete. (Moore complete, but not Moore-complete, Moore space constructed by M.E. Rudin in [6] is a subcompact space that is not the space of maximal elements of any Scott domain, even though the space is domain-representable in the light of Theorem 3.1.

An easy corollary of Theorem 3.1 is that any T_1 -space with exactly one limit point is domainrepresentable. However, a more careful examination of the proof of Theorem 2.1 allows us to show much more, namely:

Proposition 3.5 Any T₁-space with exactly one limit point is Scott-domain-representable.

Proof: Suppose (X, τ) is a T_1 -space with exactly one limit point q. Then X is actually a T_3 -space, and X is an infinite set. Let

$$P = \{\{x\} : x \in X - \{q\}\} \cup \{U : q \in U \in \tau\}$$

and let \sqsubseteq be reverse inclusion. Note that each member of *P* is both closed and open in *X* and that if $U, V \in P$ with $U \cap V \neq \emptyset$, then $U \cap V \in P$. Also note that each member of *P* is either a singleton or is an infinite open set containing *q*, and that if $U, V \in P$ and $q \in U \cup V$, then $U \cup V \in P$.

Observe that, because \sqsubseteq is reverse inclusion, a subset $I \subseteq P$ is an ideal of (P, \sqsubseteq) if and only if (1) if $U \in I$ and $U \subseteq V \in P$, then $V \in I$ and (2) if $U, V \in I$, then $U \cap V \in I$. (In other words, the ideals of (P, \sqsubseteq) are exactly the filters in P.)

Because *P* is a subcompact base for *X*, the proof of Theorem 3.1 shows that $(\mathbb{I}(P), \subseteq)$ is a domain whose subspace of maximal elements is homeomorphic to *X*. To complete this proof, suppose that $I, J, K \in \mathbb{I}(P)$ with $I, J \subseteq K$. We show that there is an ideal $L \in \mathbb{I}(P)$ that is the least upper bound of *I* and *J*. There are several cases to consider. The cases depend upon whether both, just one, or neither of *I* and *J* contain a singleton set. (Note that no ideal of *P* can contain two different singleton sets.)

First, suppose that *I* contains the singleton set $S_1 = \{x\}$ and that *J* contains the singleton set $S_2 = \{y\}$. Because $S_1, S_2 \in K$ we know that $S_1 \cap S_2 \neq \emptyset$ and therefore x = y. Note that every member of *P* containing $\{x\}$ must belong to *I*, and that every member of *I* must contain $\{x\}$. Hence $I = \{\{x\}\} \cup \{U \in P : x \in U\}$. The same applies to the ideal *J* so that I = J and we use L = I = J.

Second, suppose that exactly one of the ideals, say *I*, contains a singleton set, say $\{x\}$. For any $V \in J$, note that $\{x\}$ and *V* have a common extension in *K*, so that $x \in V$. Because $\{x\} \in I$ the facts that *I* is an ideal and $\{x\} \subseteq V \in P$ shows that $V \in I$. Consequently, $J \subseteq I$ and we have $I = \sup(I, J)$.

Third, suppose that neither *I* nor *J* contain any singleton set. Consider any $U \in I, V \in J$. Then $U, V \in K$ so that $U \cap V \in K \subseteq P$. Let $L = \{U \cap V : U \in I, V \in J\}$. We claim that *L* is an ideal of *P*. Clearly, if $W_i = U_i \cap V_i \in L$ where $U_i \in I$ and $V_i \in J$ for i = 1, 2, then $U_1 \cap U_2 \in I$ and $V_1 \cap V_2 \in J$, so that $(U_1 \cap U_2) \cap (V_1 \cap V_2) \in L$, so that $W_1 \cap W_2 \in L$. Next suppose that $U_3 \cap V_3 \in L$ with $U_3 \in I$ and $V_3 \in J$, and that $U_3 \cap V_3 \subseteq W \in P$. Because neither *I* nor *J* contains any singleton sets, $q \in U_3 \cap V_3 \subseteq W$. Then $U_3 \subseteq U_3 \cup W \in P$ so that $U_3 \cup W \in I$. Similarly $V_3 \cup W \in J$, so that $(U_3 \cup W) \cap (V_3 \cup W) \in L$. But $(U_3 \cup W) \cap (V_3 \cup W) = W$ so that $W \in L$. Therefore, $L \in \mathbb{I}(P)$. Because $X \in I$ we know that $J \subseteq L$, and analogously $I \subseteq L$. Hence *L* is an upper bound for I, J in $\mathbb{I}(P)$. It is easy to see that $L \subseteq M$ whenever $M \in \mathbb{I}(P)$ has $I, J \subseteq M$, so that in the third case, $L = \sup(I, J)$. \Box

In [3] we showed that is X is domain representable and if Y is a space obtained from X by isolating any set of points of X, then Y is also domain-representable, and we deduced that any subspace of a space of ordinals is domain-representable. The same technique shows that if X is representable using a Scott domain, then so is any space Y obtained by isolating any set of points of X. Then, as in [3] we obtain:

Proposition 3.6 Let X be any subspace of a space of ordinals. Then X is (hereditarily) Scott-domain representable.

Question 3.7 Suppose the X is hereditarily Scott-domain representable. Must X be scattered?

4 Representability of certain strongly α -favorable spaces

K. Martin [9] has proved that any domain-representable space has a strong Baire Category completeness property called *Choquet-completeness*, where a space X is Choquet complete if Player α has a winning strategy in the Choquet game Ch(X). Recall that in the Choquet game Ch(X) on a space X, Players α and β alternate moves, with β playing first by specifying a pair (U_1, x_1) with U_1 open and $x_1 \in U_1$. Player α responds by specifying an open set V_1 that is required to contain the point x_1 . Player β responds by specifying a pair (U_2, x_2) with $x_2 \in U_2 \subseteq V_1$, where x_2 might be different from x_1 , and U_2 is open. Players α and β continue their alternating choices and a play of the game is an infinite sequence $U_1, x_1, V_1, U_2, x_2, V_2, U_3, x_3, \cdots$ of moves by the two players. Player α wins that play provided $\bigcap \{U_n : n \ge 1\} \neq \emptyset$. A strategy for Player α in the game Ch(X) is a sequence of functions $\langle \sigma_n \rangle$ that tells Player α how to respond at any stage of the game: if the preceding moves in the game are $U_1, x_1, V_1, U_2, x_2, V_2, \dots, U_n, x_n$, then Player α should respond by specifying the open set $V_n = \sigma_n(U_1, x_1, V_1, U_2, x_2, V_2, \dots, U_n, x_n)$. To say that X is *Choquet complete* means that Player α has a strategy $\langle \sigma_n \rangle$ such that Player α wins every play of the game in which σ_n is used to specify α 's response set V_n for each $n \ge 1$. In this case, $\langle \sigma_n \rangle$ is said to be a *winning strategy for Player* α . Note that when applying the function σ_n at the n^{th} stage of the game, Player α has complete information about all steps of the game up to that point.

A more restrictive type of strategy for Player α in Ch(X) is a *stationary strategy*, i.e., a single function σ that associates to each pair (U,x) with U open and $x \in U$ an open set $\sigma(U,x)$ with $x \in \sigma(U,x) \subseteq U$. A stationary strategy σ for Player α is said to be a *winning strategy* if Player α wins every play $U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ of the game provided $V_k = \sigma(U_k, x_k)$ for each k, and the space X is said to be *strongly* α -*favorable* provided Player α has a winning strategy in Ch(X). Obviously, every strongly α -favorable space is Choquet-complete.

Which spaces are strongly α -favorable? Clearly, any locally compact Hausdorff space is strongly α -favorable. Among metrizable spaces, the properties of strong α -favorability and α -favorability are both

equivalent to Čech completeness. More generally, Porada [13] has shown that every Čech-complete space is strongly α -favorable.

The equivalence of Čech-completeness with the two types of α -favorability that holds in metrizable spaces does not hold in Moore spaces, as Martin and Reed have shown [11]. There are two different types of completeness in Moore spaces that are commonly studied. One is Moore completeness (often called "completeness"), a property that is equivalent to Čech-completeness and to countable Čech completeness in Moore spaces (See 3.2.1 in [1]). A second kind of completeness is Moore spaces is Rudin completeness (also called "weak completeness") that is equivalent to subcompactness and to countable subcompactness in Moore spaces (see 3.2.2 in [1]). Weak completeness in a Moore space was defined in Section 2. Both types of completeness in Moore spaces give strong α -favorability, in the light of the next lemma.

Lemma 4.1 Any countably subcompact T_3 -space is strongly α -favorable.

Proof: Let \mathcal{B} be a countably subcompact base for X. Then any countable regular filter base $\mathcal{F} \subseteq \mathcal{B}$ has $\bigcap \mathcal{F} \neq \emptyset$. Given any pair (U, x) where U is open in X and $x \in U$, let $\sigma(U, x)$ be any member $V \in \mathcal{B}$ with $x \in V \subseteq \operatorname{cl}(V) \subseteq U$. Then σ is a winning strategy for Player α in Ch(X). \Box

Some special types of domain-representability are known to imply strong α -favorability. Martin [9] has shown that if a space *X* is homeomorphic to the space of maximal elements of a Scott domain, then *X* is strongly α -favorable. (Scott domains are defined at the end of Section 3.)

We now turn to the main point of this section, investigating which strongly α -favorable spaces are domain-representable. We begin by considering strongly α -favorable spaces that have G_{δ} -diagonals. Recall that *X* has a G_{δ} -diagonal provided there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of *X* with the property that for each $x \in X$, $\bigcap \{ St(x, \mathcal{G}(n)) : n \ge 1 \} = \{x\}.$

Lemma 4.2 Suppose that X is a strongly α -favorable space with a G_{δ} -diagonal. Then there is a stationary winning strategy σ for Player α in the Choquet game Ch(X) with the property that if $U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ is any play of Ch(X) with $V_n = \sigma(U_n, x_n)$, then $|\bigcap \{V_n : n \ge 1\}| = 1$.

Proof: Let τ be any winning stationary strategy for Player α in Ch(X) and let $\mathcal{G}(n)$ be any G_{δ} -diagonal sequence for X. We may assume that $\mathcal{G}(n+1)$ refines $\mathcal{G}(n)$ and that $\mathcal{G}(1) = \{X\}$. For a nonempty open set U we write $U < \mathcal{G}(n)$ to mean that $U \subseteq G$ for some $G \in \mathcal{G}(n)$. Observe that any non-empty open set U has $U < \mathcal{G}(1)$ and that if a non-empty open set U has $U < \mathcal{G}(n)$ for each n, then U is a singleton open set.

We are now ready to define the new strategy σ . For any pair (U, x) with $x \in U$ and U open, define $\hat{U} = U$ if $U < \mathcal{G}(n)$ for each n, and otherwise let n = n(U) be the largest integer with $U < \mathcal{G}(n)$. Let \hat{U} be any open set with $x \in \hat{U} \subseteq U$ and $\hat{U} < \mathcal{G}(n+1)$. Now for any pair (U, x) with $x \in U$ and U open, define $\sigma(U, x) = \tau(\hat{U}, x)$.

Suppose that $U_1, x_1, V_1, U_2, x_2, V_2, \cdots$ is any play of Ch(X) in which $V_n = \sigma(U_n, x_n) = \tau(\hat{U}_n, x_n)$. Then $\hat{U}_1, x_1, \hat{V}_1, \hat{U}_2, x_2, V_2, \cdots$ is a play of Ch(X) with $V_n = \tau(\hat{U}_n, x_n)$ for each *n* so that $\bigcap \{V_n : n \ge 1\} \neq \emptyset$. Let $p \in \bigcap \{V_n : n \ge 1\}$. Then $p \in V_n \subseteq \sigma(U_n, x_n) = \tau(\hat{U}_n, x_n) \subseteq \hat{U}_n$. If there is any set U_n with $U_n < \mathcal{G}_k$ for all *k*, then U_n is a singleton, and so are \hat{U}_n and $V_n = \sigma(U_n, x_n)$, so that $\bigcap \{V_k : k \ge 1\}$ is a singleton, as claimed. Consider the remaining case, where no $U_n < \mathcal{G}_k$ for all *k*. Because $\mathcal{G}(1) = \{X\}$ the largest integer n_1 such that $U_1 < \mathcal{G}(n_1)$ satisfies $n_1 \ge 1$ so that $\hat{U}_1 < \mathcal{G}(n_1 + 1)$ which refines $\mathcal{G}(2)$. Because $U_2 \subseteq V_1 \subseteq \hat{U}_1$ we see that the largest integer n_2 such that $U_2 < \mathcal{G}(n_2)$ satisfies $n_2 \ge 2$, so that $\hat{U}_2 < \mathcal{G}(3)$. In general, we have $\hat{U}_n < \mathcal{G}(n+1)$. Therefore, $V_n \subseteq \hat{U}_n$ guarantees that the non-empty set $\bigcap \{V_n : n \ge 1\}$ must be a singleton, as required. \Box

Proposition 4.3 Suppose that X is a strongly α -favorable space with a G_{δ} -diagonal. Then X is domain-representable.

Proof: Let σ be a winning stationary strategy for Player α in Ch(X) that has the additional property described in Lemma 4.2. Let \mathcal{T} be the collection of all open sets of the space X and let $P = \{(V, y) : y \in V \in \mathcal{T}\}$. For $(U, x), (V, y) \in P$ define $(U, x) \sqsubseteq (V, y)$ to mean that either (U, x) = (V, y) or else $V \subseteq \sigma(U, x)$. Then (P, \sqsubseteq) is a poset, so that by Theorem 2.1, we have the ideal completion $(\mathbb{I}(P), \subseteq)$, which is known to be a domain. What remains is to show that the set of maximal members of $(\mathbb{I}(P), \subseteq)$ with the relative Scott topology is homeomorphic to the space X.

Step 1: For each $x \in X$, the collection $N(x) = \bigcup \{ \downarrow ((U,x)) : x \in U \in \mathcal{T} \}$ is an ideal of the poset (P, \sqsubseteq) . It is immediate that if $(W,z) \sqsubseteq (V,y) \in N(x)$ then $(W,z) \in N(x)$, so that it suffices to find, for any $(V_i, y_i) \in N(x)$ some $(W,x) \in P$ with $(V_i, y_i) \sqsubseteq (W,x)$. Associated with (V_i, y_i) we have some $(U_i, x) \in P$ with $(V_i, y_i) \sqsubseteq (U_i, x)$. Let $W = \sigma(U_1, x) \cap \sigma(U_2, x)$ and note that, because σ is a strategy in the Choquet game Ch(X), $x \in W$. We have $(V_i, y_i) \sqsubseteq (W, x)$ where the latter relation holds because $W \subseteq \sigma(U_i, x)$. This completes Step 1.

Step 2: For any maximal ideal *M* of *P*, there is some $x \in X$ with M = N(x), with N(x) defined as in Step 1. There are two cases to consider. The first is where *M* contains a maximal element, say (V,y), or itself. Then $M = \downarrow ((V,y))$. We know that $y \in \sigma(V,y)$ so that $(\sigma(V,y),y) \in P$, and $(V,y) \sqsubseteq (\sigma(V,y),y)$. Therefore $M = \downarrow ((V,y)) \subseteq \downarrow ((\sigma(V,y),y))$ so that maximality of *M* yields $M = \downarrow ((V,y)) = \downarrow ((\sigma(V,y),y)$. Therefore $(\sigma(V,y),y) \sqsubseteq (V,y)$ so that either $\sigma(V,y) = V$ or else $V \subseteq \sigma(V,y) \subseteq V$. In either case, we have $V = \sigma(V,y)$. Next we claim that $V = \sigma(V,y) = \{y\}$. For contradiction, suppose there is some $z \neq y$ with $z \in V = \sigma(V,y)$. Let $W = V - \{z\} = \sigma(V,y) - \{z\}$. The $y \in W$ so that $(W,y) \in P$ and $(V,y) \sqsubseteq (W,y)$. Consequently, $M = \downarrow (V,y) \subseteq \downarrow (W,y)$. Maximality of *M* shows that $M = \downarrow ((V,y)) = \downarrow ((W,y))$ so that $(W,y) \in \downarrow ((V,y))$. Therefore $(W,y) \sqsubseteq (V,y)$ so that either W = V or else $V \subseteq \sigma(W,y) \subseteq W$. In either case we have $z \in V \subseteq W \subseteq X - \{z\}$ and that is impossible. Therefore, if *M* has a maximal element (V,y), then $V = \sigma(V,y) = \{y\}$, showing that *y* is an isolated point of *X*. But then $M = \downarrow ((V,y)) = \downarrow ((\{y\}, y\}) \subseteq N(y)$ so that maximality of *M* gives M = N(y) as required to complete Step 2, provided *M* has a maximal element.

It remains to consider the case where *M* is a maximal ideal of (P, \sqsubseteq) that does not contain any maximal member of itself. Choose distinct $(U_i, x_i) \in M$ with $(U_i, x_i) \sqsubseteq (U_{i+1}, x_{i+1})$ for each $i \ge 1$. Then $x_{i+1} \in U_{i+1} \subseteq \sigma(U_i, x_i)$ so that $(U_1, x_1), \sigma(U_1, x_1), (U_2, x_2), \sigma(U_2, x_2), \cdots$ is a play of the Choquet game Ch(X) in which Player α uses the special winning strategy σ , so that from Lemma 4.2 we know that the set $\bigcap \{U_i : i \ge 1\}$ is a singleton, say $\bigcap \{U_i : i \ge 1\} = \{x\}$. For contradiction, assume some member $(V_0, y_0) \in M$ has $x \notin V_0$, i.e., $V_0 \subseteq X - \{x\}$. Because *M* is directed and has no maximal element, we may recursively find distinct $(V_i, y_i) \in M$ with

- a) $(V_0, y_0), (U_1, x_1) \sqsubseteq (V_1, y_1)$ and
- b) if $i \ge 1$ then $(V_i, y_i), (U_{i+1}, x_{i+1}) \subseteq (V_{i+1}, y_{i+1})$.

Then $(V_1, y_1), \sigma(V_1, y_1), (V_2, y_2), \sigma(V_2, y_2), \cdots$ is also a play of Ch(X) in which player α uses the special strategy σ , so that $\bigcap \{V_i : i \ge \} \neq \emptyset$. The recursion conditions give us that $V_i \subseteq U_i$ for each i so that $\emptyset \neq \bigcap \{V_i : i \ge 1\} \subseteq \bigcap \{U_i : i \ge 1\} = \{x\}$, showing that $x \in V_i$ for each $i \ge 1$. But recursion condition (a) gives $(V_0, y_0) \sqsubseteq (V_1, y_1)$ so that $x \in V_1 \subseteq \sigma(V_0, y_0) \subseteq V_0 \subseteq X - \{x\}$ and that is impossible. Therefore, if $(V, y) \in M$, then $x \in V$.

We need a bit more, namely that $x \in \sigma(V, y)$ whenever $(V, y) \in M$. Consider any $(V, y) \in M$. Because M has no maximal element, we can find $(G,t) \in M - \{(V,y)\}$ with $(V,y) \sqsubseteq (G,t)$. Therefore, $G \subseteq \sigma(V,y)$. But because $(G,t) \in M$, we know that $x \in G \subseteq \sigma(V,y)$.

To complete the proof of the second case in Step 2, consider any $(V, y) \in M$. We know that $x \in \sigma(V, y)$ so that $(\sigma(V, y), x) \in P$. Consequently, $(\sigma(V, y), x) \in N(x)$ and $(V, y) \sqsubseteq (\sigma(V, y), x)$ from the definition of \sqsubseteq . From Step 1, N(x) is an ideal of *P* so that $(V, y) \sqsubseteq (\sigma(V, y), x) \in N(x)$ yields $(V, y) \in N(x)$. Therefore $M \subseteq N(x)$ so that maximality of *M* gives M = N(x) as cl;aimed in Step 2.

Step 3: For each $x \in X$, the collection N(x), defined above, is a maximal ideal of *P*. Zorn's lemma guarantees that there is some maximal ideal *M* of *P* with $N(x) \subseteq M$, and Step 2 guarantees that M = N(y) for some $y \in X$. We claim that x = y. If not, let $H = X - \{y\}$. Then $(H, x) \in P$ so that $(H, x) \in N(x) \subseteq M = N(y)$. Therefore, some $(L, y) \in P$ has $(H, x) \sqsubseteq (L, y)$ so that either H = L or else $L \subseteq \sigma(H, x)$. In either case we have $L \subseteq H$. Because $(L, y) \in P$ we know that $y \in L \subseteq H = X - \{y\}$ and that is impossible. Therefore x = y and we have $N(x) \subseteq M = N(x)$ showing that N(x) is a maximal ideal of *P*.

Step 4: The function $h: X \to \max(\mathbb{I}(P))$ given by h(x) = N(x) is a homeomorphism from X onto $\max(\mathbb{I}(P))$. That $h[X] \subseteq \max(\mathbb{I}(P))$ is Step 2 above. That $h[X] = \max(\mathbb{I}(P))$ is Step 3 above. That h is 1-1 is proved as in Step 3. It remains to show that h is continuous and a relatively open mapping, where $\max(\mathbb{I}(P))$ carries the relative Scott topology from $\mathbb{I}(P)$.

As explained in Section 2, we know that basic neighborhoods in the subspace $\max(\mathbb{I}(P))$ have the form $\Uparrow(J(V,y)) \cap \max(\mathbb{I}(P))$ where $(V,y) \in P$ and $J(V,y) = \{I \in \mathbb{I}(P) : J(V,y) \subseteq I\} = \{I \in \mathbb{I}(P) : (V,y) \in I\}$. From $h(x) = N(x) \in \Uparrow(J(V,y))$ we know that $J(V,y) \subseteq N(x)$ so that $(V,y) \in J(V,y) \subseteq N(x)$. Consequently, for some open set U we have $x \in U$ (so that $(U,x) \in P$) and $(V,y) \sqsubseteq (U,x)$. Therefore, either (V,y) = (U,x) or else $U \subseteq \sigma(V,y)$.

In case (V, y) = (U, x), find the open set $W = \sigma(V, y)$ and note that $x \in W$ so that $(W, x) \in P$. Consider any $z \in W$. Because $W = \sigma(V, y) \subseteq \sigma(V, y)$ we have $(V, y) \sqsubseteq (W, z)$ for every $z \in W$. Therefore $J(V, y) \subseteq$ N(z). Because J(V, y) is a compact element of the domain $\mathbb{I}(P)$ we know that $J(V, y) \ll J(V, y) \subseteq N(z)$ so that $h(z) = N(z) \in \Uparrow(J(V, y))$ as required for continuity. In case $(V, y) \neq (U, x)$, then $(V, y) \sqsubseteq (U, x)$ gives $U \subseteq \sigma(V, y)$ so that for any $z \in U$ we have $(V, y) \sqsubseteq (U, z)$, showing that $(V, y) \in N(z)$ and hence that $h(z) \in \uparrow(J(V, y)) = \Uparrow(J(V, y))$. Therefore, h is continuous.

To finish the proof, we show that *h* is a relatively open mapping. Let $x \in U$ with *U* open, and consider any $y \in U$. We will show that h[U] is a relative Scott neighborhood of h(y) in the subspace max($\mathbb{I}(P)$). Because $y \in U$ we know that $(U, y) \in P$ so that J(U, y) is a compact element of $\mathbb{I}(P)$. Because $J(U, y) \subseteq N(y)$ we see that $J(U, y) \ll J(U, y) \subseteq N(y)$ so that $h(y) = N(y) \in \uparrow(J(U, y))$. We claim that $\uparrow(J(U, y)) \cap \max(\mathbb{I}(P)) \subseteq h[U]$. For suppose $M \in \uparrow(J(U, y)) \cap \max(\mathbb{I}(P))$. Then M = h(z)for some *z*. But $M = N(z) \in \uparrow(J(U, y))$ gives $J(U, y) \subseteq N(z)$ so that $(U, y) \in N(z)$. Therefore there is some $(W, z) \in P$ with $(U, y) \sqsubseteq (W, z)$. Hence either (U, y) = (W, z) or else $W \subseteq \sigma(U, y) \subseteq U$. In either case $z \in W$ implies $z \in U$, so that $M = N(z) = h(z) \in h[U]$. Therefore, *h* is open. \Box

Next we turn to a very general type of space that includes all Moore spaces. Recall that a base \mathcal{B} for *X* is a *base of countable order* if any sequence of distinct sets $B_i \in \mathcal{B}$ with $p \in B_{i+1} \subseteq B_i$ for each $i \ge 1$ must be a neighborhood base at p [16].

Proposition 4.4 Suppose that X is a T_1 -space with a base of countable order. If X is strongly α -favorable, then X is domain-representable.

Proof: The proof closely parallels the proof of Proposition 4.3. \Box

We have found that Proposition 4.3 is a surprisingly good way to recognize domain-representable spaces. We illustrate this by showing that the Nagata plane is domain representable. It can also be

used to show that the Sorgenfrey line, the Michael line, the Moore plane, and Heath's V-space are domain-representable.

Example 4.5 The Nagata plane is a non-metrizable, separable, first-countable M_1 -space that is domain-representable.

Proof: The points of N are all ordered pairs (x, y) of real numbers with 0 < x < 1 and $y \ge 0$. Points $(x, y) \in N$ with y > 0 have their usual neighborhoods, and basic neighborhoods of points $(p, 0) \in N$ have the form

$$N(p,k) = \{(p,0)\} \cup \{(x,y \in N : y < k - (k^2 - (x - p))^{\frac{1}{2}}, |x - p| < \frac{1}{k}\},\$$

i.e., the point (p,0) together with all points below a circle with center (p,k) and radius k. See Example 9.2 in [4] for details.

To show that *N* is domain-representable, note that *N* has a G_{δ} -diagonal because the weaker Euclidean topology has one. Now suppose that a pair (U,a) is given, with $a \in U$ and *U* open in *N*. If a = (x,y) with y > 0, compute $\varepsilon(U,a) = \sup\{\varepsilon > 0 : B(a,\varepsilon) \subseteq U\}$ where $B(a,\varepsilon)$ denotes the usual Euclidean ball centered at *a* with radius ε . Then define $\sigma(U,a) = B(a, \frac{\varepsilon(U,a)}{2})$. If a = (p,0), let k(U,a) be the first positive integer *k* with $N(p,k) \subseteq U$ and define $\sigma(U,a) = N(p,2k(U,a))$. Then any play $U_1,a_1,V_1,U_2,a_2,V_2,\cdots$ of the Choquet game in which $V_n = \sigma(U_n,a_n)$ has $\bigcap\{V_n : n \ge 1\}$ equal to a singleton, so that σ is a stationary winning strategy for Player α .

We close with another application of Proposition 4.3. In an e-mail to the authors, K. Martin wrote that there does not seem to be a published example of a domain-representable space that is Hausdorff but not regular. We use Proposition 4.3 to construct an example of that type.¹

Example 4.6 There is a Hausdorff space X that is not regular and that is homeomorphic to $\max(Q)$ for some domain (Q, \sqsubseteq) . In addition, the usual space of rational numbers is a closed subspace of $\max(Q)$.

Proof: Let \mathbb{Q} and \mathbb{P} denote the sets of rational and irrational numbers, respectively. To obtain *X*, we modify the topology of the usual space \mathbb{R} of real numbers by making each $x \in \mathbb{P}$ have basic neighborhoods of the form $(x - \varepsilon, x + \varepsilon) \cap \mathbb{P}$, while letting any $x \in \mathbb{Q}$ have its usual open neighborhoods. Then the set \mathbb{P} becomes a dense open subspace of *X* and the subspace of rational numbers becomes a closed subspace of *X*. (As subspaces of *X*, both \mathbb{P} and \mathbb{Q} carry their usual Euclidean topology.)

Because the closure of a neighborhood $(a,b) \cap \mathbb{P}$ of an irrational number must contain all rational numbers in (a,b), X is not regular. However, X is Hausdorff and has a G_{δ} -diagonal because the usual topology on the set of real numbers has those properties and is contained in the new topology we defined for X.

The usual space \mathbb{P} of irrational numbers is an open subspace of *X*. Because \mathbb{P} is a completely metrizable space, Player α has a stationary winning strategy τ in the Choquet game $Ch(\mathbb{P})$. Now suppose that we are given a pair (U,x) where *U* is open in *X* and $x \in U$. If $x \in \mathbb{Q}$, let $\sigma(U,x)$ be any interval (a,b) with $x \in (a,b) \subseteq [a,b] \subseteq U$, and if $x \in \mathbb{P}$ we let $\sigma(U,x) = \tau(U \cap \mathbb{P},x)$. It is easy to see that σ is a stationary winning strategy for Ch(X), so that Proposition 4.3 applies to show that *X* is domain-representable. \Box

¹A more interesting example would be a domain Q with a countable domain-basis where $\max(Q)$ is Hausdorff but not regular. We do not have such an example.

5 Complete quasi-developable spaces

Recall that a space X is *quasi-developable* (see [2]) if there is a sequence $\langle \mathcal{G}(n) \rangle$ of collections of open sets with the property that for each $x \in X$, the collection $\{St(x, \mathcal{G}(n)) : n \ge 1 \text{ and } x \in \bigcup \mathcal{G}(n)\}$ is a local base at x. (If each $\mathcal{G}(n)$ is an open cover of X, then X is developable.) A quasi-development $\langle \mathcal{G}(n) \rangle$ for X is *complete* if, for every sequence $n_1 < n_2 < \cdots$, $\bigcap \{M_i : i < \omega\} \neq \emptyset$ whenever each M_i is nonempty, closed, $M_{i+1} \subseteq M_i$, and for some $G_i \in \mathcal{G}(n_i), M_i \subseteq G_i$. This is a direct generalization of completeness (also called "Moore-completeness") in Moore spaces. The main theorem in this section, Theorem 5.4, generalizes the theorem of Martin and Reed [11] that any complete Moore space is domain-representable. We begin with a pair of technical lemmas that allow us to overcome certain bad behavior of arbitrary quasi-developments.

Lemma 5.1 Suppose that $\langle \mathcal{H}(n) \rangle$ is a complete quasi-development for a space X. Then there is a complete quasi-development $\langle \mathcal{G}(n) \rangle$ for X such that for each even integer $k \ge 1$, $\mathcal{G}(k) = \{\{x\} : x \text{ is an isolated point of } X\}$.

Proof: For any odd integer 2j - 1, let $\mathcal{G}(2j - 1) = \mathcal{H}(j)$ and for any even integer 2k, define $\mathcal{G}(2k)$ be the collection of all isolated singletons in *X*. Then $\langle \mathcal{G}(n) \rangle$ is a complete quasi-development for *X*. \Box

Lemma 5.2 Let $\langle \mathcal{G}(n) \rangle$ be a complete quasi-development for a T_1 -space X as described in Lemma 5.1. If m is any integer and if y is a point of an open set U, then there is some n > m with $y \in St(y, \mathcal{G}(n)) \subseteq U$.

Proof: If *y* is an isolated point of *X*, we let *n* be any even integer greater than *m*. Hence assume that *y* is not an isolated point of *X*. Then there is some *k* with $y \in St(y, \mathcal{G}(k)) \subseteq U$. If k > m the proof is complete, so assume $k \leq m$. Then the set $V = U \cap \bigcap \{St(y, \mathcal{G}(i)) : i \leq m \text{ and } y \in St(y, \mathcal{G}(i))\}$ is an open set containing *y*. Because *y* is not isolated, we may choose $z \in V - \{y\}$. Then there is some *n* with $y \in St(y, \mathcal{G}(n)) \subseteq V - \{z\}$. Necessarily n > m and $St(y, \mathcal{G}(n) \subseteq St(y, \mathcal{G}(k) \subseteq U$ as required. \Box

Next, starting with $\langle \mathcal{G}(n) \rangle$, a complete quasi-development for *X* as described in Lemmas 5.1 and 5.2, we define special new sets C(x,k) whenever $x \in \bigcup \mathcal{G}(k)$. For each $k \ge 1$ and each $x \in \bigcup \mathcal{G}(k)$ first choose a set g(x,k) with $x \in g(x,k) \in \mathcal{G}(k)$. Now we recursively define sets C(x,k) for $x \in \bigcup \mathcal{G}(k)$. For $x \in \bigcup \mathcal{G}(1)$ let C(x,1) = g(x,1). Now suppose that the sets C(x,n) have been defined whenever $x \in \bigcup \mathcal{G}(n)$ with $n \le k$. For each $x \in \bigcup \mathcal{G}(k+1)$ let

$$C(x,k+1) = \left(\bigcap \{C(x,i) : i \le k \text{ and } x \in \bigcup \mathcal{G}(i)\}\right) \cap g(x,k+1).$$

Finally, define $P = \{(C(x,k),k) : k \ge 1 \text{ and } x \in \bigcup \mathcal{G}(k)\}$. (It may seem redundant to write (C(x,k),k) rather than just C(x,k). However it is conceivable that, as sets, C(x,j) = C(y,k) for some x, y, j, k with $j \ne k$, and it will be important to keep the "levels" of *P* distinct.) For any $p \in P$, $\pi_i(p)$ will denote the i^{th} coordinate of the ordered pair *p*.

Partially order *P* by defining $(C(x_1, n_1), n_1) \sqsubseteq (C(x_2, n_2), n_2)$ to mean either that the ordered pairs are identical or else that $n_1 < n_2$ and $cl(C(x_2, n_2)) \subseteq C(x_1, n_1)$. Apply Theorem 2.1 to find a domain $\mathbb{I}(P), \subseteq$) that contains (an order-isomorphic copy of) (P, \sqsubseteq) as its set of compact elements. The key to identifying the maximal elements of $\mathbb{I}(P)$ will be our next lemma.

Lemma 5.3 Suppose that $\langle G(n) \rangle$ is a complete quasi-development for the space X as in Lemmas 5.1 and 5.2, and suppose that $\langle (C(x_i, n_i), n_i) \rangle$ is a sequence of distinct members of P with $(C(x_i, n_i), n_i) \subseteq (C(x_{i+1}, n_{i+1}), n_{i+1})$. Then:

1) $\bigcap \{C(x_i, n_i) : i \ge 1\} \neq \emptyset;$

- 2) if $y \in \bigcap \{C(x_i, n_i) : i \ge 1\}$ then the sequence $\langle x_i \rangle$ converges to y;
- 3) $|\bigcap \{C(x_i, n_i) : i \ge 1\}| = 1$ so that $\bigcap \{C(x_i, n_i) : i \ge 1\} = \{p\}$ where p is as in (2).

Proof: Because sequential limits are unique in a T_2 -space, (2) implies (3). Because $cl(C(x_{i+1}, n_{i+1})) \subseteq C(x_i, n_i) \subseteq g(x_i, n_i) \in \mathcal{G}(n_i)$ for each *i*, completeness of *X* guarantees that (1) holds. Therefore it remains only to prove assertion (2).

To prove assertion (2), we will show that y is a limit point of $\langle x_i \rangle$. With obvious modifications, the same argument will show that y is a limit point of every subsequence of $\langle x_i \rangle$, from which assertion (2) follows immediately.

For contradiction, suppose that *y* is not a limit point of the sequence $\langle x_i \rangle$. Then there is some *J* such that $y \notin cl(\{x_i : i > J\})$ so there is some k_1 with $y \in St(y, \mathcal{G}(k_1)) \subseteq X - \{x_i : i > J\}$. For each i > J let $M_i = cl(\{x_j : j > i\})$. Then

$$M_{i+1} \subseteq \operatorname{cl}(C(x_{i+1}, n_{i+1})) \subseteq C(x_i, n_i) \subseteq g(x_i, n_i) \in \mathcal{G}(n_i)$$

so that completeness of X guarantees that $\bigcap \{M_i : i > J\} \neq \emptyset$. Fix any point $s \in \bigcap \{M_i : i > J\}$. Then s is a limit point of $\langle x_i \rangle$, while y is not, so that $s \neq y$. According to Lemma 5.2, there is some $k_2 > k_1$ with $s \in St(s, \mathcal{G}(k_2)) \subseteq X - \{y\}$. Because s is covered by the collection $\mathcal{G}(k_2)$ the set $g(s, k_2) \in \mathcal{G}(k_2)$ is defined.

Because *s* is a limit point of $\langle x_i \rangle$ there is some *j* with $n_j > k_2$ and $x_j \in g(s,k_2)$. Observe that $x_j \in g(s,k_2) \in \mathcal{G}(k_2)$ so that the set $C(x_j,k_2)$ is defined. Therefore $n_j > k_2$ yields $C(x_j,n_j) \subseteq C(x_j,k_2)$ so that we have

$$y \in C(x_j, n_j) \subseteq C(x_j, k_2) \subseteq g(x_j, k_2) \in \mathcal{G}(k_2).$$

Because $s \in M_j \subseteq C(x_j, k_2) \subseteq g(x_j, k_2) \in \mathcal{G}(k_2)$ we know that *s* and *y* belong to the same member $g(x_j, k_2) \in \mathcal{G}(k_2)$, showing that $y \in St(s, \mathcal{G}(k_2)) \subseteq X - \{y\}$, and that is impossible. Hence *y* is a limit point of $\langle x_i \rangle$. As noted at the beginning of this proof, the same argument shows that *y* is a limit point of every subsequence of $\langle x_i \rangle$ so that $\langle x_i \rangle$ converges to *y*, as required. \Box

Now we are ready to prove the main theorem in this section, namely:

Theorem 5.4 Suppose that X is a complete quasi-developable T_3 space. Then X is domain representable.

Proof: We use the notation developed earlier in this section. For any $x \in X$ let $D(x) = \{(C(x,k),k) : k \ge 1, x \in \bigcup \mathcal{G}(k)\}$ Because X is regular, Lemma 5.2 shows that D(x) is a directed subset of (P, \sqsubseteq) . Let $N(x) = \{(C(y,n),n) \in P : \text{for some } (C(x,k),k) \in D(x), (C(y,n),n) \sqsubseteq (C(x,k),k)\}$. Then N(x) is an ideal of (P, \sqsubseteq) .

Consider any maximal ideal *M* if (P, \sqsubseteq) . If *M* contains a maximal element of itself, then clearly $\bigcap \{\pi_1(m) : m \in M\} \neq \emptyset$, and if *M* does not contain any maximal element of itself, choose any strictly increasing sequence $(C(x_1, n_1), n_1) \sqsubseteq (C(x_2, n_2), n_2) \sqsubseteq \cdots$ in *M*. Lemma 5.3 shows that there is a unique point $z \in \bigcap \{C(x_i, n_i) : i \ge 1\}$ and that the sequence $\langle x_i \rangle$ converges to *z*. If some member (C(y, j), j) of *M* has $z \notin C(y, j)$ then directness of *M* would allow us to recursively find a strictly increasing sequence of other elements of $(C(w_k), m_k) \in M$ such that $(C(y, j), j), (C(x_i, n_i), n_i) \sqsubseteq (C(w_i, m_i), m_i)$. But then $\bigcap \{C(w_i, m_i) : i \ge 1\} = \emptyset$, contradicting Lemma 5.3. In any case, therefore, some $y \in X$ has $y \in \bigcap \{C(z, j) : (C(z, j), j) \in M\}$ so that $M \subseteq N(y)$.

Next consider any ideal N(z) for $z \in X$. Use Zorn's lemma to find a maximal ideal M of P with $N(z) \subseteq M$. From the previous paragraph we know that for some $y \in X$, M = N(y) so that $N(z) \subseteq M = N(y)$. Hence z = y and we have N(z) = M is maximal.

Define a function $h: X \to \max(\mathbb{I}(P))$ by the rule that h(z) = N(z). The previous paragraphs show that *h* is 1-1 and onto.

Recall from Section 2 that basic neighborhoods in the Scott topology on $\mathbb{I}(P)$ have the form $\uparrow (J(C(y,i),i))$ which is the same as $\uparrow (J(C(y,i),i)$ because each (C(y,i),i) is a compact element of $\mathbb{I}(P)$. We now verify that *h* is continuous. Suppose that $h(z) = N(z) \in \uparrow (J(C(y,i),i))$. Then $(C(y,i),i) \in J(C(y,i),i) \subseteq N(z)$ so that, by definition of N(z), there is some $(C(z,j),j) \in P$ with $(C(y,i),i) \sqsubseteq (C(z,j),j)$. Therefore, from the definition of \sqsubseteq , we know that $z \in C(z,j) \subseteq C(y,i)$. Consider any $w \in C(z,j)$. Lemma 5.2 shows that we can find some $k > \max(i,j)$ with

$$w \in \operatorname{St}(w, \mathcal{G}(k)) \subseteq \operatorname{cl}(\operatorname{St}(w, \mathcal{G}(k))) \subseteq C(z, j) \subseteq C(y, i).$$

Because $w \in St(w, \mathcal{G}(k))$, the set C(w,k) is defined and we have $C(w,k) \subseteq g(w,k) \subseteq St(w, \mathcal{G}(k))$ so that $cl(C(w,k)) \subseteq cl(St(w, \mathcal{G}(k))) \subseteq C(y,i)$. Because i < k we have $(C(y,i),i) \sqsubseteq (C(w,k),k)$ so that, from the definition of N(w), we know that $(C(y,i),i) \in N(w)$. That gives $J(C(y,i),i) \subseteq N(w)$ because N(w) is an ideal, so that $h(w) = N(w) \in \uparrow (J(C(y,i),i))$ as required for continuity.

To show that *h* is an open mapping from *X* to $\max(\mathbb{I}(P))$, suppose *U* is open in *X* and $x \in U$. We will show that h[U] contains a relative Scott neighborhood of h(x) in $\max(\mathbb{I}(P))$. Because *X* is regular and quasi-developable, there is an integer *i* with $x \in \operatorname{St}(x, \mathcal{G}(i)) \subseteq \operatorname{cl}(\operatorname{St}(x, \mathcal{G}(i))) \subseteq U$. Then C(x,i) is defined and $C(x,i) \subseteq \operatorname{St}(x, \mathcal{G}(i))$, $(C(x,i),i) \in P$, and $\uparrow (J(C(x,i),i) \cap \max(\mathbb{I}(P)))$ is a relative Scott neighborhood of h(x). We claim that $\uparrow (J(C(x,i),i) \cap \max(\mathbb{I}(P)) \subseteq h[U]$. For let $M \in \uparrow (C(x,i),i)) \cap \max(\mathbb{I}(P))$. Then for some $y \in X$, M = h(y) = N(y). We have $(C(x,i),i) \in J(C(x,i),i) \subseteq M = N(y)$ so that for some $(C(y,j),j) \in P$ we have $(C(x,i),i) \sqsubseteq (C(y,j),j)$. Therefore $y \in C(y,j) \subseteq C(x,i) \subseteq \operatorname{St}(x, \mathcal{G}(i)) \subseteq U$, showing that $M = h(y) \in h[U]$ as required. \Box

Remark 5.5 It is conceivable that Theorem 5.4 might be a consequence of Theorem 3.1 because we do not have an example of a complete quasi-developable space that is not subcompact. However, subcompactness is not equivalent to completeness among quasi-developable spaces: the Michael line is quasi-developable and subcompact (see 2.4.5 in [1]), but is not a complete quasi-developable space because, as proved in [2], any paracompact complete quasi-developable space is metrizable.

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