Scott Domain Representability of a Class of Generalized Ordered Spaces

Kevin W. Duke* and David Lutzer†

Draft of July 25, 2007

Abstract

Many important topological examples (the Sorgenfrey line, the Michael line) belong to the class of GO-spaces constructed on the usual set $\mathbb{R}$ of real numbers. In this paper we show that every GO-space constructed on the real line, and more generally, any GO-space constructed on a locally compact LOTS, is Scott-domain representable, i.e., is homeomorphic to the space of maximal elements of some Scott domain with the Scott topology.

MR Classifications: primary = 54F05; secondary = 54D35,54D45,54D80, 06F30

Key words and phrases: Scott domain, Scott topology, dcpo, domain-representable space, generalized ordered space, GO-space.

1 Introduction

A topological space X is a *Baire space* if every intersection of countably many dense, open sets is dense. The Baire space property does not behave well under topological operations, and in the 1960s several authors (Choquet, deGroot, and Oxtoby, for example) described topological properties now called *Choquet completeness, subcompactness, and pseudo-completeness* that are stronger than the Baire space property and are well-behaved under the product operation. Such properties were studied in [1] and have come to be thought of as being strong completeness properties.

More recently, topologists have borrowed a property called *domain representability* from theoretical computer science and have come to be see it as a kind of completeness property related to the Baire property. A space X is *domain representable* if X is homeomorphic to the space of maximal elements of some domain, topologized with the relative Scott topology. (Definitions appear in Section 2.) Research has shown that domain representability fits into the hierarchy of strong completeness properties mentioned above: every subcompact regular space is domain representable [3] and every domain representable space is Choquet complete [4].

Examples in [2] show that the Sorgenfrey line and the Michael line are both domain-representable. Later, [3] showed that any regular space X that has a G_δ-diagonal and in which player α has a stationary

*College of William and Mary, Williamsburg, VA 23187-8795. E-mail = kwduke@wm.edu. This paper is part of the first author's undergraduate thesis.

†College of William & Mary, Williamsburg, VA 23187. E-mail = lutzer@math.wm.edu

winning strategy in the strong Choquet game $Ch(X)$ must be domain representable. It follows from that last result that any generalized ordered space (GO-space) constructed on the real line must be domain representable.

Scott domains are a very special type of domain, and being Scott-domain representable (i.e., being homeomorphic to the subspace of maximal elements of a Scott-domain with the Scott topology) is a property much stronger than domain representability. In this paper, we use direct constructions to show that any GO-space constructed on the real line is Scott-domain representable. In fact, in Theorem 3.2 we prove even more, namely that any generalized ordered space whose underlying LOTS topology is locally compact must be representable as the space of maximal elements of some Scott domain. (We will refer to such spaces as "GO-spaces constructed from a locally compact linearly ordered space.") This more general result applies to, for example, any GO-space constructed on a set of ordinals, and to any GO-space constructed on the Alexandroff double arrow, as well as to GO-spaces constructed on $\mathbb{R}$. Examples such as the Sorgenfrey and Michael lines show that GO-spaces constructed from a locally compact LOTS can be very pathological.

We want to thank the referee whose suggestions significantly improved an earlier draft of this paper.

2 Basic definitions

Let $(P, \sqsubseteq)$ be a partially ordered set (poset). Following [5], we say that a subset $D \subseteq P$ is *directed* if for each finite $F \subseteq D$, some $d \in D$ has $x \sqsubseteq d$ for all $x \in F$. This is equivalent to the assertion that $D \neq \emptyset$ and if $d_1, d_2 \in D$, then some $d_3 \in D$ has $d_1, d_2 \sqsubseteq d_3$. To avoid confusion, we will often be redundant and speak of "non-empty directed sets." The poset $(P, \sqsubseteq)$ is said to be a *directed-complete partial order* (dcpo) if $\sup(D) \in P$ for every nonempty directed $D \subseteq P$[1]. There is an auxiliary relation $\ll$ associated with $\sqsubseteq$. We say that $a \ll b$ if for every nonempty directed $D \subseteq P$, if $b \sqsubseteq \sup(D)$ then some $d \in D$ has $a \sqsubseteq d$. For $p, q \in P$, we use the notations:

$$\Uparrow(p) := \{q \in P : p \ll q\} \text{ and } \Downarrow(p) := \{q \in P : q \ll p\}$$

If for each $p \in P$ the set $\Downarrow(p)$ is nonempty and directed and has $p = \sup(\Downarrow(p))$ then we say that $(P, \sqsubseteq)$ is a *continuous* poset. A continuous poset that is also a dcpo is called a *domain*, and a domain P is a *Scott domain* if it has the extra property that if $p_1, p_2 \in P$ have $p_1, p_2 \sqsubseteq p_3$ for some $p_3 \in P$ then $\sup\{p_1, p_2\} \in P$. If P is a domain, then the collection $\{\Uparrow(p) : p \in P\}$ is a basis for a topology on P called the *Scott topology*. The set of maximal elements of a domain $(P, \sqsubseteq)$ is denoted by $\max(P)$. To say that a topological space X is *(Scott) domain representable* means that there is a (Scott) domain $(P, \sqsubseteq)$ such that X is homeomorphic to the subspace $\max(P)$ with the relative Scott topology. As noted above, domain representability lies between deGroot's subcompactness and what is now called "strong Choquet completeness" ([3], [4]).

A *generalized ordered space* (GO-space) is a triple $(X, <, \sigma)$ where $<$ is a linear ordering of X and σ is a Hausdorff topology on X that has a base of order-convex [2] open sets. We will say that σ is a *GO-topology* on the set $(X, <)$. The usual open interval topology λ of the order $<$ has $\lambda \subseteq \sigma$ for each GO-topology σ on $(X, <)$. The Sorgenfrey and Michael lines, and subspaces of spaces of ordinals, are familiar GO-spaces that have been important counterexamples in topology. Clearly, even if one starts with a very nice linearly ordered space such as the usual space of real numbers, GO-spaces constructed on it can be pathological.

[1] By $\sup(D)$ we mean an element of P that is the least of all upper bounds of D.

[2] A set $C \subseteq X$ is *order convex* if $x \in C$ whenever $a \leq x \leq b$ with $a, b \in C$.

In this paper we reserve the symbols $\mathbb{R}, \mathbb{P}$, and $\mathbb{Q}$ for the sets of real, irrational, and rational numbers, respectively.

3 The Main Theorem

We begin with some notation that we will use throughout our paper.

Definition 3.1 *For any GO-space $(X,<,\sigma)$ let λ denote the usual open interval topology of the order $<$. Let:*

a) $I := \{x \in X : \{x\} \in \sigma\}$

b) $R := \{x \in X - I : [x, \rightarrow [\ \in \sigma\}$

c) $L := \{x \in X - I :\ ] \leftarrow, x] \in \sigma\}$

d) $E := X - (R \cup L \cup I)$

e) $\mathcal{K} := \{[a,b] : a < b \ \&\ [a,b] \text{ is compact in the topology } \lambda\}$

f) $Y := X \times \{0,1\}$ *with the lexicographic order and* $\pi : Y \to X$ *is defined by* $\pi((x,i)) = x$.

The symbol $<$ will denote both the given order on X and the lexicographic order of Y. Context will make it clear which order is being used. For example, when we write $(a,i) < (b,j)$ the reader will see that $<$ is being used for the lexicographic ordering on Y.

Theorem 3.2 *Let $(X,<)$ be a linearly ordered set whose usual open interval topology λ is locally compact. Let $(X,<,\sigma)$ be any GO space constructed on $(X,<)$. Then (X,σ) is Scott-domain representable.*

Outline of Proof: Our proof has two main steps. In the first (Proposition 3.3) , we consider the special case where a point x is isolated in (X,σ) if and only if it is isolated in (X,λ), In the second we use a general construction (Proposition 3.12) to show how to obtain the general case of Theorem 3.2 from Proposition 3.3. Details follow the proof of 3.12.

Proposition 3.3 *Suppose $(X,<,\tau)$ is a GO-space constructed on a linearly ordered set whose open interval topology λ is locally compact. Also suppose that a point x is isolated in τ if and only if it is isolated in λ. Then (X,τ) is Scott-domain representable.*

To prove Proposition 3.3 we will construct a poset $\mathcal{P}$ whose members are closed, bounded intervals of the lexicographically ordered set $Y := X \times \{0,1\}$ with reverse inclusion as the partial order. We will show that

a) $\mathcal{P}$ is a dcpo (Lemma 3.6),

b) $\mathcal{P}$ is a Scott poset (Lemma 3.7),

c) $\mathcal{P}$ is continuous (Lemma 3.9), and

d) there is a homeomorphism from (X,τ) onto $\max(\mathcal{P})$, where the range space carries the relative Scott topology (Lemma 3.11).

The symbol $(*)$ will denote the following assumption:

$(*)$ (X,λ) is locally compact and $\{x\} \in \tau$ if and only if $\{x\} \in \lambda$.

Definition 3.4 *Suppose $[u,v] \in \mathcal{K}$. Then we define the following notations for certain subsets of $Y := X \times \{0,1\}$:*

a) $I(u,v) := [(u,0),(v,1)]$

b) *if $u \in R$,* $J(u,v) := [(u,1),(v,1)]$

c) *if $v \in L$,* $K(u,v) := [(u,0),(v,0)]$

d) *if $u \in R$ and $v \in L$,* $M(u,v) := [(u,1),(v,0)]$

e) *for any $x \in X$,* $D(x) := \{(x,0),(x,1)\}$

f) *for any $x \in R$,* $S(x) := \{(x,1)\}$

g) *for any $x \in L$,* $S(x) := \{(x,0)\}$.

We will say that a subset of Y is of *type I* if it can be written as $I(u,v)$ for some $[u,v] \in \mathcal{K}$. Sets of *type J, type K, type M, type D*, and *type S* are analogously defined. For notational convenience we will use the phrase "sets of the form $U(s,t)$ where $U \in \{I,J,K,M\}$" to mean that $U(s,t)$ is one of the sets $I(s,t), J(s,t), K(s,t), M(s,t)$.

Definition 3.5 $\mathcal{P} := \{U(s,t) : [s,t] \in \mathcal{K} \;\&\; U \in \{I,J,K,M\}\} \cup \{D(x) : x \in X\} \cup \{S(x) : x \in R \cup L\}$ *and $\sqsubseteq$ is reverse inclusion.*

Lemma 3.6 *The poset $(\mathcal{P},\sqsubseteq)$ is a dcpo.*

Proof: Suppose $\mathcal{E}$ is a nonempty directed subset of $\mathcal{P}$. Then each member of $\mathcal{E}$ is a compact convex subset of Y, so that $\bigcap \mathcal{E}$ is also a nonempty compact convex subset of Y. Write $\bigcap \mathcal{E} = [(a,i),(b,j)]$ with $a \leq b$. If we can show that $[(a,i),(b,j)] \in \mathcal{P}$ then Lemma 3.6 will be proved.

There are two cases. If $a = b$ then $\emptyset \neq \bigcap \mathcal{E} \subseteq D(a)$. If $\bigcap \mathcal{E} = D(a)$ we are done. In the remaining case $\bigcap \mathcal{E} = \{(a,i)\}$ for some i. Consider the case where $i = 0$. Then some $E \in \mathcal{E}$ has $(a,0) \in E$ and $(a,1) \notin E$. This could happen in only two ways: if $E = K(c,a)$ or $E = M(c,a)$ with $a \in L$, or if $E = S(a)$ with $a \in L$. In either case, $\bigcap \mathcal{E} = \{(a,0)\} \in \mathcal{P}$. The case where $i = 1$ is analogous.

Now consider the case where $a < b$. Each member of $\mathcal{E}$ contains $[(a,i),(b,j)]$ so that $[a,b] \in \mathcal{K}$. If $i = 0$ and $j = 1$, then $[(a,i),(b,j)] = I(a,b) \in \mathcal{P}$. If $i = 0$ and $j = 0$, then some $E \in \mathcal{E}$ contains $(b,0)$ but not $(b,1)$ so that E must be of type K or M. In either case, $b \in L$ so that $[(a,0),(b,0)] = K(a,b) \in \mathcal{P}$. The case where $i = 1$ is analogous and leads to the conclusion that $\bigcap \mathcal{E} = J(a,b)$ or $M(a,b)$. $\Box$

Lemma 3.7 *If $p,q \in \mathcal{P}$ and $p \cap q \neq \emptyset$, then $p \cap q \in \mathcal{P}$. Hence $(\mathcal{P},\sqsubseteq)$ is a Scott poset.*

Proof: If one of p,q contains the other, there is nothing to prove. Therefore, if one of p,q has the form $S(x)$ for some $x \in R \cup L$, then $p \cap q \in \mathcal{P}$.

<u>Case 1</u>: Suppose that one of p,q has the form $D(x)$, say $p = D(x) = \{(x,0),(x,1)\}$. Then $p \cap q \subseteq D(x)$. If $p \cap q = D(x)$ there is nothing to prove, so assume that $p \cap q = \{(x,i)\}$. In case $i = 0$, q contains $(x,0)$ but not $(x,1)$ so that $x \in L$ and therefore $p \cap q = \{(x,0)\} \in \mathcal{P}$. The case where $i = 1$ is analogous.

For the rest of the proof, we assume that $\pi[p]$ and $\pi[q]$ each have at least two points so that $p = [(a,i),(b,j)]$ and $q = [(c,s),(d,t)]$ for some choices of $a,b,c,d \in X$ with $a < b$ and $c < d$, and $i,j,s,t \in \{0,1\}$. Without loss of generality we may assume $(a,i) \leq (c,s)$. Given that $p \cap q \neq \emptyset$ and that neither of p,q contains the other, it follows that $(a,i) \leq (c,s) \leq (b,j) < (d,t)$ so that $p \cap q = [(c,s),(b,j)]$. Because $[a,b],[c,d] \in \mathcal{K}$ it follows from $p \cap q \neq \emptyset$ that $[c,b] \in \mathcal{K}$.

Case 2: Suppose $p = I(a,b)$. Then $i = 0$ and $j = 1$. If $s = 0$ then $p \cap q = [(c,0),(b,1)] = I(c,b) \in \mathcal{P}$ and if $s = 1$ then $q = [(c,1),(d,t)]$ is of type J or M, so that $c \in R$. Then $p \cap q = [(c,1),(b,1)] = J(c,b) \in \mathcal{P}$.

Case 3: Suppose $p = J(a,b)$. Then $i = 1, j = 1$, and $a \in R$ so that $p \cap q = [(c,s),(b,1)]$. If $s = 0$ then $p \cap q = I(c,b) \in \mathcal{P}$ and if $s = 1$ then $q = [(c,1),(d,t)] \in \mathcal{P}$ gives $c \in R$. But then $p \cap q = [(c,1),(d,1)] = J(c,d) \in \mathcal{P}$.

Case 4: Suppose $p = K(a,b) = [(a,1),(b,0)]$ with $a \in R, b \in L$. If $s = 0$ then $p \cap q = [(c,0),(b,1)] = J(c,b) \in \mathcal{P}$. If $s = 1$, then $q = [(c,1),(d,t)] \in \mathcal{P}$ yields $c \in R$ so that $p \cap q = [(c,1),(b,0)] = M(c,b) \in \mathcal{P}$ because $c \in R$ and $b \in L$.

Case 5: Suppose $p = M(a,b) = [(a,1),(b,0)]$ with $a \in R, b \in L$. Then $p \cap q = [(c,s),(b,0)]$. If $s = 0$ then $p \cap q = K(c,b) \in \mathcal{P}$. If $s = 1$, then $q = [(c,1),(d,t)] \in \mathcal{P}$ which forces $c \in R$. Then $p \cap q = [(c,1),(b,0)] = M(c,b) \in \mathcal{P}$ because $c \in R$ and $b \in L$. $\square$

Lemma 3.8 *For $p,q \in \mathcal{P}$, $p \ll q$ if and only if $q \subseteq \mathrm{Int}_Y(p)$.*

Proof: Suppose $q \subseteq \mathrm{Int}_Y(p)$ and $q \sqsubseteq \sup(\mathcal{E})$ for some directed subset of $\mathcal{P}$. Then $\bigcap \mathcal{E} \subseteq q \subseteq \mathrm{Int}_Y(p)$ so that compactness of the members of $\mathcal{E}$ gives some $E \in \mathcal{E}$ with $E \subseteq \mathrm{Int}_Y(p)$ and therefore $p \sqsubseteq E$ as required to show $p \ll q$.

Conversely, suppose $p \ll q$. Then $p \sqsubseteq q$ so that $q \subseteq p$. Because p and q are compact convex subsets of Y, the only way that $q \subseteq \mathrm{Int}_Y(p)$ can fail is

(**) some endpoint of p belongs to q and is not an interior point of p.

The rest of the proof is a case-by-case analysis of the possible shapes of p and q.

Case 1: Suppose $p = I(a,b) = [(a,0),(b,1)]$. By (**), one of $(a,0),(b,1)$ belongs to q and is not interior to p. Consider the sub-case where the problematic endpoint is $(a,0)$, the other case being analogous. Because $(a,0) \in p - \mathrm{Int}_Y(p)$ we know that a is not the left endpoint of X and that $a = \sup_X(\,]\leftarrow,a[\,)$. By (**) we know that $(a,0)$ is the left endpoint of q.

In case $|q| = 1$ we have $q = \{(a,0)\}$ and then we know that $a \in L$. Because (X,λ) is locally compact at a there is a compact λ-neighborhood of a of the form $[v,w]$ with $v \leq a \leq w$. Because $a = \sup_X(\,]\leftarrow,a[\,)$, a has no immediate predecessor in X. Hence $v < a$ and $]v,a[\, \neq \emptyset$, and for each $e \in\,]v,a[\,$, $[e,a] \in \mathcal{K}$ so that the set $K(e,a) = [(e,0),(a,0)] \in \mathcal{P}$. Then $\mathcal{E} := \{K(e,a) : v < e < a\}$ is a directed subset of $\mathcal{P}$ with $p \ll q = \sup(\mathcal{E})$ so that some $K(e,a) \in \mathcal{E}$ must have $p \sqsubseteq K(e,a)$ and that is impossible because $(e,0) \in K(e,a) - p$.

In case $|q| > 1$ then $D(a) \subseteq q$. With $v < a$ as above, if $v < e < a$ then $I(e,a) \in \mathcal{P}$. We let $\mathcal{E} = \{I(e,a) : v < e < a\}$ and obtain a directed set with $p \ll q \sqsubseteq D(a) = \sup(\mathcal{E})$ and that is impossible because $(e,0) \in I(e,a) - p$ for each $e \in\,]v,a[\,$.

Case 2: Suppose $p = J(a,b) = [(a,1),(b,1)]$ where $a \in R$. Because $(a,1) \in \mathrm{Int}_Y(p)$, the endpoint of p to which (**) refers must be $(b,1)$. Note that $(b,1)$ is the right endpoint of q and consequently $b \neq \max(X)$ and $b = \inf_X(\,]\leftarrow,b[\,)$. Using local compactness of (X,λ) find $v \leq b \leq w$ in X such that

$[v,w]$ is a compact neighborhood of b. Because b has no immediate successor in X, we know that $b < w$ and $b = \inf_X(\,]b,w[\,)$. For each $e \in\ ]b,w[\,,\ [b,e] \in \mathcal{K}$.

We know that $(b,1) \in q$. If, in addition, $(b,0) \in q$, we let $\mathcal{E} := \{I(b,e) : b < e < w\}$ and obtain a directed subset of $\mathcal{P}$ with $\bigcap \mathcal{E} = D(b) \subseteq q$ so that $p \ll q \sqsubseteq \sup(\mathcal{E})$, showing that some $I(b,e) \in \mathcal{E}$ has $p \sqsubseteq I(b,e)$ and that is impossible because $I(b,e) \not\subseteq p$. In case $(b,0) \notin q$, then $q = S(b) = \{(b,1)\}$ showing that $b \in R$. Then $J(b,e) \in \mathcal{P}$ for each $e \in\ ]b,w[$ so that $\mathcal{E} := \{J(b,e) : b < e < w\}$ is a directed subset of $\mathcal{P}$ with $p \ll q \sqsubseteq \sup(\mathcal{E})$ and that is impossible because no $J(b,e) \in \mathcal{E}$ has $p \sqsubseteq J(b,e)$.

Case 3: Suppose $p = K(a,b) = [(a,0),(b,0)]$ where $b \in L$. Then the problematic endpoint of p mentioned in (**) must be $(a,0)$, and this case is analogous to Case 2.

Case 4: Suppose $p = M(a,b) = [(a,1),(b,0)]$ for $a \in R, b \in L$. This case cannot occur, being incompatible with (**) because the set p is open in Y so that $p \subseteq q$ yields $p \subseteq \mathrm{Int}_Y(q)$.

Case 5: Suppose that $p = D(x) = \{(x,0),(x,1)\}$. Suppose that $(x,0)$ is the endpoint of p mentioned in (**). Because $p \ll q$ yields $q \subseteq p$ we know that either $q = p = D(x)$ or else $q = \{(x,0)\}$. In the first case, because $(x,0)$ is not interior to $p = D(x)$ we know that $x \neq \min(X)$ and $x = \sup_X(\,]\leftarrow,x[\,)$. Local compactness gives some $w > x$ such that $[x,w]$ is compact and then $\mathcal{E} := \{I(x,e) : x < e < w\}$ is a directed subset of $\mathcal{P}$ with $p \ll q \sqsubseteq \sup(\mathcal{E})$ so that $p \sqsubseteq I(x,e)$ for some e with $x < e < w$, and that is impossible. In the remaining sub-case, $q = \{(x,0)\}$ so that $x \in L$. Local compactness gives $v < x$ such that $x = \sup_X(\,]v,x[\,)$ and $[v,x]$ is compact. Then $K(e,x) = [(e,0),(x,0)] \in \mathcal{P}$ so that $\mathcal{E} := \{K(e,x) : v < e < x\}$ is a directed subset of $\mathcal{P}$ with $p \ll q = \sup(\mathcal{E})$ and hence $p \sqsubseteq K(e,x)$ for some $e \in\ ]v,x[\,$, which is impossible. In case $(x,1)$ is the problematic endpoint mentioned in (**), the argument is analogous.

Case 6: Suppose $p = S(x)$ where $x \in R \cup L$. Then $p \ll q$ gives $\emptyset \neq q \subseteq p$ so that $q = S(x) = p$. Suppose $x \in R$. Then $S(x) = \{(x,1)\}$. Because $(x,1) \notin \mathrm{Int}_Y(p)$ we know that $x \neq \max(X)$ and $x = \inf_X(\,]x,\rightarrow[\,)$. Local compactness gives some $w > x$ such that $[x,w]$ is compact. If $x < e < w$, $J(x,e) = [(x,1),(e,0)] \in \mathcal{P}$. Then $\mathcal{E} := \{J(x,e) : x < e < w\}$ is directed and has $p \ll q \sqsubseteq \sup(\mathcal{E})$ which leads to the same contradiction as is previous cases. The case where $p = S(x)$ with $x \in L$ is analogous. □

Lemma 3.9 $(\mathcal{P},\sqsubseteq)$ *is a continuous poset.*

Proof: Fix $q \in \mathcal{P}$. From Lemma 3.8 we know that $\Downarrow(q) = \{p \in \mathcal{P} : p \ll q\} = \{p \in \mathcal{P} : q \subseteq \mathrm{Int}_Y(p)\}$. Fix $p_1, p_2 \in \Downarrow(q)$. Then $p_i \ll q$ gives $\emptyset \neq q \subseteq \mathrm{Int}_Y(p_i)$ so that Lemma 3.7 shows $p_1 \cap p_2 \in \mathcal{P}$. Also $q \subseteq \mathrm{Int}_Y(p_1) \cap \mathrm{Int}_Y(p_2) = \mathrm{Int}_Y(p_1 \cap p_2)$ so that $p_1 \cap p_2 \in \Downarrow(q)$.

Next we must show that $\Downarrow(q) \neq \emptyset$. We do this by examining cases.

Case 1: Suppose $q = U(a,b)$ where $U \in \{I,J,K,M\}$. We know that $[a,b] \in \mathcal{K}$. Local compactness of (X,λ) gives points $r \leq a \leq s$ and $u \leq b \leq v$ such that $[r,s]$ and $[t,u]$ are compact λ-neighborhoods of a and b respectively. Note that if a is the left endpoint of X, or if a has an immediate predecessor in X, then $r = a$ is possible. In all other cases, $r < a$. Analogous assertions apply to b and v. In any case, $[r,v]$ is λ-compact and contains a in its λ-interior. Then $I(r,v) \in \mathcal{P}$ and $q = U(a,b) \subseteq \mathrm{Int}_Y(I(r,v))$ so that $I(r,v) \in \Downarrow(q)$.

Case 2: Suppose $q = D(x)$ or $q = S(x)$. Local compactness gives points $a \leq x \leq b$ with $x \in \mathrm{Int}_X([a,b]) \subseteq [a,b] \in \mathcal{K}$ Then $I(a,b) \in \mathcal{P}$ and $q \subseteq \mathrm{Int}_Y(I(a,b))$ so that $I(a,b) \in \Downarrow(q)$.

We now know that $\Downarrow(q)$ is nonempty and directed. To complete the proof, we show that $q = \sup \Downarrow(q)$. If q is open in Y, then Lemma 3.8 shows that $q \ll q$ so that $q \in \Downarrow(q)$ and hence $q = \sup(\Downarrow(q))$. Therefore, assume q is not open in Y. Then some endpoint of q is not in the interior of q. In case

$q = U(a,b)$ for $U \in \{I,J,K,M\}$, write $q = [(a,i),(b,j)]$. Local compactness gives $c \leq a < b \leq d$ such that $[a,b] \subseteq \mathrm{Int}_X([c,d])$ and $[c,d] \in \mathcal{K}$.

Because q is not open in Y, one endpoint of q is not interior to q. Suppose $(a,i) \notin \mathrm{Int}_Y(q)$. Then $i = 0, a \neq \min(X)$, and $a = \sup_X(\,]\leftarrow,a[\,)$. Because $a \in \mathrm{Int}_\lambda([c,d])$, $c < a$ and $]c,a[\; \neq \emptyset$. If $j = 0$ then $b \in L$ and $K(e,b) = [(e,0),(b,0)] \in \mathcal{P}$, and $U(a,b) \subseteq [(a,0),(b,0)] \subseteq \mathrm{Int}_Y(K(e,b))$ for each $e \in \;]c,a[$ so that $\{K(e,b) : c < e < a\} \subseteq \Downarrow(q)$ has supremum q. Hence $q = \sup(\Downarrow(q))$. In case $j = 1$, then $q = [(a,0),(b,1)]$. If $b = \max(X)$ or if b has an immediate successor in X then $I(e,b) = [(e,0),(b,1)] \in \mathcal{P}$ and $q \subseteq \mathrm{Int}_Y(I(e,b))$ so that $I(e,b) \in \Downarrow(q)$ for each $e \in \;]c,a[$. Because $q = \sup(\{I(e,b) : c < e < a\})$ we have $q = \sup(\Downarrow(q))$ as required. If, on the other hand, $b \neq \max(X)$ and $b = \inf_X(\,]b \rightarrow [\,)$ then the compact set $[r,v]$ found above has $r < a < b < v$ so that $q \subseteq \mathrm{Int}_Y(I(d,e)) \subseteq I(d,e)$ for each $d \in \;]r,a[$ and $e \in \;]b,v[$. Therefore the collection $\mathcal{E} = \{I(d,e) : r < d < a \;\&\; b < e < v\}$ is directed and has supremum q. Because $\mathcal{E} \subseteq \Downarrow(q)$ we have $q = \sup(\Downarrow(q))$. This completes the case where $i = 0$. The case where $i = 1$ is analogous. □

Lemma 3.10 *The set of maximal elements of* $(\mathcal{P},\sqsubseteq)$ *is* $\max(\mathcal{P}) = \{S(x) : x \in R \cup L\} \cup \{D(x) : x \in X - (R \cup L)\}$. □

Lemma 3.11 *Under assumption (*), define* $h : X \rightarrow \max(\mathcal{P})$ *by the rule that* $h(x) = S(x)$ *if* $x \in R \cup L$ *and* $h(x) = D(x)$ *if* $x \in X - (R \cup L)$. *Then h is a homeomorphism from X onto* $\max(\mathcal{P})$.

Proof: Clearly h is a bijection. To show that h is continuous, fix $x \in X$ and suppose $h(x) \in \Uparrow(p)$. We must find an open set U with $x \in U$ and $h[U] \subseteq \Uparrow(p)$. In case $x \in I$, use $U = \{x\}$. Now assume that $x \in X - I = R \cup L \cup (E - I)$. The proof analyzes eighteen separate cases, corresponding to $x \in R, x \in L$, and $x \in E - I$ and $p = U(a,b)$ for $U \in \{I,J,K,M\}$ and $p = D(x), S(x)$. Luckily there is great similarity among the cases.

<u>Case 1</u>: Suppose $x \in R$. If $p = I(a,b) = [(a,0),(b,1)]$, then $h(x) = \{(x,1)\} \subseteq \mathrm{Int}_Y[(a,0),(b,1)]$ so that $a \leq x \leq b$. If $a \leq x < b$ use $U = [x,b)$. The case where $x = b$ cannot occur because $(x,1) = (b,1) \in \mathrm{Int}_Y[(a,0),(b,1)]$ would imply that x is either the right hand end point of X or that x has an immediate successor in X, and that (combined with $x \in R$) would yield $x \in I$ contrary to $x \in R \subseteq X - I$. The same argument applies to the case where $p = J(a,b) = [(a,1),(b,1)]$ with $a \in R$. In case $p = K(a,b)$ or $p = M(a,b)$ we see that $a \leq x < b$ and we use $U = [x,b)$. In case $p = D(y) = \{(y,0),(y,1)\}$ we see that $h(x) = \{(x,1)\} \subseteq \mathrm{Int}_Y(D(y))$ forces $y = x$ and either $x = \max(X)$ or else x has an immediate successor in X. Either case combines with $x \in R$ to show that $x \in I$, which is impossible. The case where $p = S(y)$ is analogous.

<u>Case 2</u>: The case where $x \in L$ is analogous to Case 1.

<u>Case 3</u>. Suppose $x \in X - (R \cup L \cup I)$. Then $h(x) = D(x) = \{(x,0),(x,1)\}$ and we have

> (***) The point x is not an endpoint of X and basic neighborhoods of x have the form $]r,s[$ where $]r,x[\; \neq \emptyset \neq \;]x,s[$.

If $p = I(a,b) = [(a,0),(b,1)]$ then $a \leq x \leq b$. Because $h(x) \subseteq \mathrm{Int}_Y(p)$, assertion (***) shows that $a = x$ and $b = x$ are both impossible, so $a < x < b$ and we use $U = \;]a,b[$. In case $p = J(a,b) = [(a,1),(b,1)]$, $h(x) = D(x)$ shows that $a < x \leq b$, and (***) shows that $x = b$ cannot occur, so we use $U = \;]a,b[$. The cases where $p = K(a,b)$ and $p = M(a,b)$ are analogous and we use $U = \;]a,b[$. Clearly $p = S(y)$ cannot have $h(x) = D(x) \subseteq S(y)$ so $p = S(y)$ cannot occur, and the case where $p = D(y)$ is excluded by (***).

We next show that h is an open mapping. To do this we suppose that U is open in X and $x \in U$, and we show that $h[U]$ is a neighborhood of $h(x)$ in the relative Scott topology on the subspace $\max(\mathcal{P})$ by finding some $p \in \mathcal{P}$ with $h(x) \in \Uparrow(p) \cap \max(\mathcal{P}) \subseteq h[U]$. This also requires a case-by-case analysis.

Case 4: Suppose $x \in I$. In the light of (*), there are three possibilities:

i) $x = \min(X)$ and some $b \in X$ has $x < b$ and $]x,b[\ = \emptyset$;

ii) $x = \max(X)$ and some $a \in X$ has $a < x$ and $]a,x[\ = \emptyset$;

iii) every neighborhood of x contains some set $]a,b[$ with $a < x < b$ and $]a,x[\ = \emptyset =\ ]x,b[$.

In each of the three cases, let $p = D(x)$. Then p is open in Y and $\{h(x)\} = \Uparrow(p) \cap \max(\mathcal{P})$.

Case 5: Suppose that $x \in R$. Then x is not the right endpoint of X and basic neighborhoods of x have the form $[x,b[$ where $\emptyset \neq\ ]x,b[$, and $h(x) = S(x) = \{(x,1)\}$. Given an open subset U of X with $x \in U$, find $[x,b[\ \subseteq [x,b] \subseteq [x,c[\ \subseteq U$ and $[x,b] \in \mathcal{K}$. Because $x \in R$, the set $J(x,b) = [(x,1),(b,1)] \in \mathcal{P}$ and $h(x) \in \Uparrow(p) \cap \max(\mathcal{P})$. Furthermore, if $h(y) \in \Uparrow(p) \cap \max(\mathcal{P})$, then $x \leq y \leq b < c$ so that $y \in [x,c) \subseteq U$, showing that $h(y) \in h[U]$ as required.

Case 6: The case where $x \in L$ is analogous to Case 5.

Case 7: Suppose that $x \in X - (R \cup L \cup I)$. Then x is not an endpoint of X and basic neighborhoods of x have the form $]a,b[$ where $a < x < b$ and neither $]a,x[$ nor $]x,b[$ is empty. Given $x \in U$ use local compactness of (X,λ) to find a,b,c,d with $c < a < x < b < d$ and $]c,d[\ \subseteq U$ and $[c,d] \in \mathcal{K}$. Let $p = I(a,b) \in \mathcal{P}$. Then $h(x) \in \Uparrow(p) \cap \max(\mathcal{P}) \subseteq h[U]$, as required. □

Proposition 3.12 *Suppose that a topological space (X,σ) is Scott-domain representable. Let $I \subseteq X$ and let $\hat{\sigma}$ be the topology having $\sigma \cup \{\{x\} : x \in I\}$ as a base of open sets. Then $(X,\hat{\sigma})$ is also Scott-domain representable.*

Proof: The proof is exactly the same as the proof of Proposition 2.1 in [2], once we note that if the construction in that proof begins with a Scott domain (rather than just a domain) then it produces a Scott-domain (rather than just a domain). □

Proof of Theorem 3.2 Suppose that $(X,\sigma,<)$ is any GO-space for which λ, the usual open interval topology of $<$, is locally compact. Let $I_1 = \{x \in X : \{x\} \in \tau - \lambda\}$. Let I, R, L, and E be as defined in (3.1) and let τ be the topology on X having the following base of open sets:

$$\{\{x\} : x \in I - I_1\} \cup \{[x,b[\ : x \in R, x < b\} \cup \{\,]a,x] : x \in L\} \cup \{\,]a,b[\ : a < x < b, x \in I_1 \cup (X - (I \cup R \cup L))\}.$$

Then τ is a GO-topology on X and $(X,\tau,<)$ satisfies the hypotheses of Proposition 3.3, so that (X,τ) is Scott-domain representable. The topology obtained from τ by isolating all points of the set I_1 is exactly the topology σ so that Proposition 3.12 applies to show that (X,σ) is also Scott-domain representable. This completes the proof of Theorem 3.2. □

Example 3.13 *Each of the following GO-spaces is Scott-domain representable:*

a) the Michael line and the Sorgenfrey line, and more generally any GO-space constructed on the real line $\mathbb{R}$;

b) any GO space constructed on the lexicographically ordered set $Y = \mathbb{R} \times \{0,1\}$;

c) any subspace of any ordinal space $[0,\kappa[$;

d) any GO-space constructed on a compact Souslin space. [3]

Question 3.14 *Can the hypothesis of local compactness in Theorem 3.2 be weakened to "the usual open interval topology λ on X is Čech-complete"? In particular, is the irrational Sorgenfrey line (= the set of all irrational numbers topologized with basic open neighborhoods of the form $[a,b[$) Scott-domain representable? If the usual open interval topology λ on X is Čech-complete, is (X,λ) Scott-domain representable?*

References

[1] Aarts, J., and Lutzer, D., Completeness properties designed for recognizing Baire spaces, *Dissertationes Math.* 114(1974), 1-44.

[2] Bennett, H. and Lutzer, D., Domain representable spaces, *Fundamenta Mathematicae*, 189(2006), 255-268.

[3] Bennett, H. and Lutzer, D., Domain representability of certain complete spaces. *Houston J. Math.*, to appear.

[4] Martin, K., Topological games in domain theory, *Topology and its Applications* 129(2003), 177-186.

[5] Mislove, M., Topology, domain theory, and theoretical computer science, *Topology and its Applications* 89(1998), 3-59.

[3] A *Souslin space* is a linearly ordered set that, in its open interval topology, is not separable and yet has countable cellularity. Whether such things exist is axiom-sensitive.